Assignment 6
Due by class time Tuesday, October 4

1. Consider the discrete 2-dimensional vector space consisting of the corners of a unit square. The entire vector space consists of just the four vectors \([0, 0], [0, 1], [1, 0],\) and \([1, 1]\). Vectors in this space are not normalized, and are not all of the same length — in fact, only the basis vectors \([1, 0]\) and \([0, 1]\) are of length 1 — but this does not matter for our purposes. Let’s call this a 2-dimensional binary space. Furthermore, suppose we have two such 2-dimensional binary spaces, called \(A\) and \(B\) (which you can think of as two different squares).

(a) Write down all of the tensor product vectors that can be formed from a vector in \(A\) and a vector in \(B\). That is, list all vectors \(v\) such that \(v = v_1 \otimes v_2\), where \(v_1 \in A\) and \(v_2 \in B\). Since \(A\) and \(B\) each have \(2^2 = 4\) elements, you should have a total of \(4 \cdot 4 = 16\) vectors in your list.

(b) How many distinct vectors are there in your list? Which ones are basis vectors for the tensor product space \(A \otimes B\)? How many basis vectors are there?

(c) The tensor product space \(A \otimes B\) is of dimension \(2 \cdot 2 = 4\), which means there are a total of \(2^4 = 16\) binary vectors in this space (corresponding to the 16 corners of a 4-dimensional hypercube). List all of the vectors in this space that do not appear in your list for part (a). How many such vectors are there?

2. Write a Python program called \(\text{binaryVectors}(n)\) that takes an integer \(n \geq 1\) as input and prints out all \(2^n\) binary vectors of dimension \(n\), in order, one vector per line. For example, if \(n = 2\), your program should print out \([0, 0], [0, 1], [1, 0],\) and \([1, 1]\).

3. Now let the dimensionality of vector space \(A\) be any integer \(m \geq 1\) and the dimensionality of vector space \(B\) be any integer \(n \geq 1\). Write a program called \(\text{showTensors}(m, n, \text{verbose})\) that takes \(m\) and \(n\) as input parameters and does what you did in Problem 1. That is, your program should determine all of the distinct separable and entangled (non-separable) tensor product vectors in \(A \otimes B\), and print them out, one vector per line, with the basis vectors highlighted in some way. It should also report the dimensionality of \(A \otimes B\), the total number of vectors in \(A \otimes B\), the number of separable vectors, and the number of entangled vectors.

The third input parameter (\(\text{verbose}\)) should be of type boolean, and should control the amount of output produced by the program. When \(\text{verbose}\) is False, the individual vectors should not be printed, and only the dimension of \(A \otimes B\), the total number of vectors in \(A \otimes B\), and the total number of separable and entangled vectors, should be reported. When \(\text{verbose}\) is True, all information should be reported.

IMPORTANT: You should include comments at the top of your code that clearly explain how to run your programs, as well as a separate demo program that tests \(\text{binaryVectors}\) and \(\text{showTensors}\) on some sample inputs.

For example, the next page shows sample output for the case of \(m = 3\) and \(n = 2\):
>>> showTensors(3, 2, True)
tensor product space is 6-dimensional
22 distinct separable tensor product vectors out of 64 possible
[0, 0, 0, 0, 0, 0]
[0, 0, 0, 0, 0, 1] <-- basis vector
[0, 0, 0, 0, 1, 0] <-- basis vector
[0, 0, 0, 0, 1, 1]
...
[1, 1, 1, 1, 1, 1]
42 entangled (non-separable) tensor product vectors out of 64 possible
[0, 0, 0, 1, 1, 0]
[0, 0, 0, 1, 1, 1]
[0, 0, 1, 0, 0, 1]
[0, 0, 1, 0, 1, 1]
...
[1, 1, 1, 1, 1, 0]

>>> showTensors(3, 2, False)
tensor product space is 6-dimensional
22 distinct separable tensor product vectors out of 64 possible
42 entangled (non-separable) tensor product vectors out of 64 possible

4. What happens to the relative proportion of separable and entangled vectors in $A \otimes B$ as the dimensionality of $A$ and $B$ increases? Use your program to fill in the table below for the (moderate) values of $m$ and $n$ shown. Beware of larger values!

<table>
<thead>
<tr>
<th>$m \otimes n$</th>
<th>dimensionality</th>
<th>total # of vectors</th>
<th># of separable</th>
<th># of entangled</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \otimes 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 \otimes 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3 \otimes 2$</td>
<td>6</td>
<td>64</td>
<td>22</td>
<td>42</td>
</tr>
<tr>
<td>$2 \otimes 3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$3 \otimes 3$</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>$4 \otimes 4$</td>
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</tr>
<tr>
<td>$4 \otimes 5$</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$5 \otimes 5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>