How to Look at Mathematical Objects

Interview with Mark Green by Philip Ording

In recalling how he came to photograph the mathematical objects, Man Ray stated flatly that he "didn't understand a thing, but the shapes were so unusual, as revolutionary as anything that is being done today in painting or in sculpture."[1] Mark Green, Professor of Mathematics Emeritus and a founding director of the Institute for Pure and Applied Mathematics at the University of California, Los Angeles, has devoted his career to investigating fundamental questions of geometry, questions whose origins trace back to the very surfaces illustrated by the models that served as Man Ray's subjects. This interview probes the kind of thinking that brought these forms into existence and their place in the history of mathematical models, from the Platonic solids of ancient Greece to the mathematical models used by physicists to describe the shape of the universe.

1 Man Ray, 1961 Interview.

What do you see when you look at these objects? Do you see mathematics? Do you see equations?
It depends on which model it is. There are ones for which the mathematics jumps out at you and for other ones it's much more just the beauty of the shape or the interest of the shape that is striking. It is not easy to describe a geometric object using words. The model of the cyclide of Dupin [fig. 84], which is a surface I had studied at some point, has a lot of mathematics that you can see rather quickly. The curves drawn on the plaster surface represent the so-called lines of curvature,[2] and it's interesting that they are all circles. Another property of this surface is that, despite having a hole in the middle, no matter how you cut it with a plane or a sphere you can only cut it into at most two pieces. Other properties—for example, that it is conformally equivalent to the torus—are also visual though harder to describe in nontechnical terms. It's a very beautiful thing mathematically. Looking at some of the models of minimal surfaces—shapes whose surface area is minimized with respect to some conditions—there's also a certain feel to that shape, a tension, which derives from the geometric property that defines it.[3]

When you think about a mathematical surface, like the ones you just described, what do you have in mind? Does your mind's eye hold an image resembling these models? Or is it something completely else?
A lot of what mathematicians who do geometry, particularly algebraic geometry, need to do is to visualize these surfaces not the way they appear here. René Descartes, and independently Pierre de Fermat, had the radical idea to describe geometric objects by

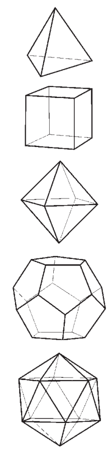
2 Think of the surface from the point of view of a golfer on a rolling, treeless green. Typically, the green will bend away from the golfer by different degrees depending on what direction he or she is looking. The directions in which the bending or curvature is maximized or minimized are called the principal directions. A line of curvature is a path in the surface that at all times travels in a principal direction. For example, the lines of curvature of a cylinder consist of the straight lines in the cylinder that are parallel to the axis of the cylinder and the circles forming the circumference of the cylinder.

3 See fig. 61 for an illustration of a minimal surface known as Enneper's minimal surface. Another way to model examples of minimal surfaces is to dip a bent wire into soapy water. The film stretching across the wire when it is lifted out of the water will approximate a minimal surface.

[fig. 84] ***Mathematical Objects,* 1934–35. Gelatin silver print**

equations. Once you do that, the entire language of algebra is available for describing geometry. On the other hand, features of the geometry are sometimes hidden—to see all of them one needs to add the square root of –1 to the usual numbers.[4] This added numerical dimension of the imaginary numbers, as they're called, corresponds to an extra spatial dimension. Many things we study are naturally four-dimensional objects. Of course, we cannot, as mathematicians, see in four dimensions, but we can visualize some things indirectly, and we kind of have to.

4 The square root of –1 is usually denoted by the symbol *i*. For example, the equation $1 = z^4$ has two real number solutions $z = -1$ and $z = 1$ and two "hidden" imaginary solutions $z = -i$ and $z = i$.



You mention the beauty or interest of the shape of these objects. What draws you to some objects of study more than others? Do mathematical objects have an emotional capacity?
I think what attracts me is really the inner harmony and the often very subtle mathematical properties that they're displaying. The universe in general, the more you understand it the more in awe you are of many different parts of it, and often these shapes are a part of that. If string theory turns out to be correct, then in very, very tiny scales certain of these algebraic surfaces will have been shown to be the basis for most of the fundamental properties of matter.

Is one of the models that Man Ray photographed relevant to string theory?
Indirectly, yes. The Kummer surfaces [fig. 66] that Man Ray photographed belong to a class of algebraic surfaces called K3 surfaces.[5] This class has been further generalized to a family of six-dimensional objects called Calabi-Yau spaces. According to string theory, there are supposed to be six curled-up dimensions in addition to the four dimensions of space-time that exist in the world, and these extra dimensions are occupied by Calabi-Yau spaces. At the time Man Ray was photographing, this was certainly very far from people's minds. But the idea that geometric things were fundamental to the universe has been around for a very long time.

5 For an example and more detail about K3 surfaces, see "A Kummer Surface," p. 100.

At least as early as the ancient Greek era.
Plato was fascinated by geometry, especially with the regular solids—the Greeks knew that there are exactly five: the cube, the triangular pyramid, the octahedron, the dodecahedron, and the icosahedron [fig. 85]. In the *Timaeus,* the dialogue in which Plato gives his version of the origin of the universe, he speculated that the four "elements" out of which the world was thought to be made—fire, air, earth, and water—were composed of bodies shaped like one of these—for example, earth was composed of cubes. The one regular solid that was left over was subsumed by Aristotle's fifth element, "aether," a substance that was conveniently to be found only in the celestial realm. Aristotle derived some properties of matter from the belief that everything was made of triangles. Now we would, of course, consider these incorrect derivations of properties,

[fig. 85] **The five Platonic solids: tetrahedron, cube, octahedron, dodecahedron, icosahedron** but the idea that geometry lay behind fundamental properties of the universe has come back, big time, over the years, for example in the theory of general relativity and string theory where it is quite fundamental. I consider it a great irony that while the universe that we all live in is not directly made of triangles as far as we know, almost all the things we construct in the virtual world in the computer are made of triangles. So, perhaps Plato is somewhere feeling very fulfilled about having been right.

It is surprising to hear that you see these models as relevant to research today. The impression I've gotten from seeing them gather dust in various mathematics libraries and departments is that they are obsolete. Have you ever used them in your own research or teaching?

When I was young, I used to spend some time as a visitor in the math department at Harvard, and I was sharing an office with a bunch of these models, and they do gather dust. I don't think I've used them in a course. Today, one can draw them on the computer and animate their motion so that you can see them reasonably well without physically rendering them. So in that sense, the technology is obsolete, but the objects themselves have become, if anything, more fundamental.

If physical models no longer serve a role in the professional training of a mathematician, how are students expected to develop their geometric intuition? As graduate students we became familiar with particular images here and there, but it wasn't systematic in any way. We were never advised to learn computer graphics programming, for example.

Certain courses, like differential geometry, have traditionally served this role. My experience of differential geometry was through a very abstract formalization, which was the popular way to learn it when I was a graduate student at Princeton. When I went to Berkeley, where Shiing-Shen Chern was, the much more traditional presentation involving curves and surfaces in Euclidean three-dimensional space was the norm.[6] And I got to teach that course, and it had a lasting impact on me in terms of how I could see things and think about things. Now, I think the use of geometric visualization in graph theory, or the mathematical study of networks, has become very important. For example, in bioinformatics you will see many, many pictures that are just ways of visualizing interrelationships between many, many things there, whether it is the reconstructions of phylogenetic trees, the microarray data relating different genes to different tumors or different types of cell cycle. . .

6 Shiing-Shen Chern (1911–2004) was a leading figure in the field of differential geometry, which applies concepts of calculus to studying problems of shape. His work has had lasting impact in both pure mathematics and theoretical physics.

So, would you say visualization has become a discipline in its own right?

Well, it is a discipline in its own right. Of course it is used in many different ways, but, yes, there are visualization centers. It's unquestionable that the ability to visualize certain things has allowed discovery, and when you can't do it, you've hindered your discovery. I think in some ways visualization is more alive and well

in applied math. When you have data that really sits in thousands of dimensions, you can't visualize it initially. You have to figure out a way to reduce the dimensionality, or find some properties in an automatic way. On the pure side, a group at MIT recently achieved a sort of visualization of a complex form of mathematical symmetry, known as the exceptional simple Lie algebra E8.[7]

7 Somewhat surprisingly, this achievement was picked up by the mainstream media. The *New York Times* characterized E8 as an object that "describes the symmetries of a 57-dimensional object that can in essence be rotated in 248 ways without changing its appearance." Kenneth Chang, "The Scientific Promise of Perfect Symmetry," *New York Times*, March 20, 2007.

Man Ray is rather explicit about his not having any particular interest or expertise in mathematics. But an awareness of mathematical questions seems to have found its way into his subconscious. He told an interviewer, "One day I had a dream I had discovered the solution of the squaring of the circle. How to transform a square into a circle. I made a circle from string and then I pulled on it with four fingers to make four corners and it became a square. Mathematically it's not possible. Except: I won't justify my dream through logic! It must remain mysterious."[8]

Early in my career, I was one of the people my department knew would be willing to talk to people who called about geometry, and some very nice guy who—I think he was working at an aerospace company—had thought he had trisected an angle.[9] I tried to explain what the question meant. That there are these certain rigid rules, and if you broke the rules, it wasn't hard to trisect an angle. But if you didn't break them, then it was just known to be theoretically impossible. At some point he said to me, "But I've been working on it so long." I felt really terrible. I think when mathematicians say that something is impossible, there's this certain kind of person who really wants to rise to the challenge, and, of course, sometimes the rules are very artificial.

8 Irmeline Lebeer, "Man Ray Fautographe," *L'Art vivant*, no. 44 (November 1973), 24; translation by the author. The problem of "squaring the circle" is to construct, by means of (mathematically idealized) compass and straightedge, a square that is equal in area to a given circle. In 1882, the challenge was proven to be impossible.

9 Another geometry challenge from the Ancients, trisecting (dividing equally in three) an arbitrary angle by compass and straightedge was proven to be impossible in 1837.

Are there mathematical fields or points in their historical development of mathematics that from your perspective function in ways similar to the avant garde or Surrealism?

The advent of truly abstract mathematics in the early twentieth century created a very different way of looking at things and a different sense of what was important. It brought about a newfound sense of freedom. Being really original was valued in Surrealism perhaps more than in almost any other movement. Overall, in mathematics we value originality greatly, but at certain times and certain places it's been especially valued. I think some of the more uncharted areas of mathematics now, like machine learning or complex systems, have more the feeling of Surrealism. The ability to build a machine to learn to do important things yet without being able to know how the computer does it and what it was exactly that it learned, these have a little bit the flavor of Surrealism. Cantor's theory of infinity is probably another good analogy to Surrealism.[10] The idea that there are distinct levels of infinity, that, for example, the numbers corresponding to the points along a number line are more numerous than the counting numbers 1, 2, 3, . . . At first sight, it's so counterintuitive.

10 A leading mathematician of the early modern era, Georg Cantor (1845–1918) believed that the "essence of mathematics lies entirely in its freedom."