The Ozsváth-Szabó and Rasmussen Concordance Invariants are not Equal

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THE OZSVÁTH-SZABÓ AND RASMUSSEN CONCORDANCE INVARIANTS ARE NOT EQUAL

By Matthew Hedden and Philip Ording

Abstract. In this paper we present several counterexamples to Rasmussen's conjecture that the concordance invariant coming from Khovanov homology is equal to twice the invariant coming from Ozsváth-Szabó Floer homology. The counterexamples are twisted Whitehead doubles of $(2, 2n+1)$ torus knots.

1. Introduction. In [18] Ozsváth and Szabó defined a smooth concordance invariant, denoted $\tau(K)$, whose value for the (p, q) torus knot provided a new proof of Milnor's famous conjecture on the unknotting number of torus knots. Rasmussen independently discovered this invariant in his thesis, [25]. Milnor's conjecture has a long history in gauge theory, and its original proof is due to Kronheimer and Mrowka, [9]. Recently, however, Rasmussen [26] discovered another smooth concordance invariant whose value for torus knots proves the conjecture. Denoted $s(K)$, the invariant is defined using a refinement, due to Lee [10], of the purely combinatorial knot (co)homology theory introduced by Khovanov [7]. Rasmussen's proof of the Milnor conjecture using s is the first proof which avoids the analytical machinery of gauge theory. It was noted immediately that the two invariants share several formal properties (e.g. an inequality relating the invariants of knots which differ by a crossing change) which in turn imply that they agree (or more precisely, that $s(K)$ and $2\tau(K)$ agree) for many knots. For instance, $s(K) = 2\tau(K)$ for the following families of knots:

(1) Torus knots: $s(K) = 2\tau(K) = 2g(K)$ where $g(K)$ denotes the Seifert genus of K. This is due to Rasmussen [26] for s and Ozsváth and Szabó [21] for τ.

(2) Alternating knots: $s(K) = 2\tau(K) = \sigma(K)$ where $\sigma(K)$ is the classical Murasugi-Trotter signature of K. This is due to Lee [10] for s, and Ozsváth and Szabó [17] for τ.

(3) Strongly quasipositive knots, in particular positive knots: $s(K) = 2\tau(K) = 2g(K)$. This is due to Livingston, [11]. See also [28].

(4) Quasipositive knots: $s(K) = 2\tau(K) = 2g_4(K)$, where $g_4(K)$ denotes the smooth slice genus of K. This follows from work of Plamenevskaya [23] for τ and from Plamenevskaya [24] and Shumakovitch [30] for s. See also [5].

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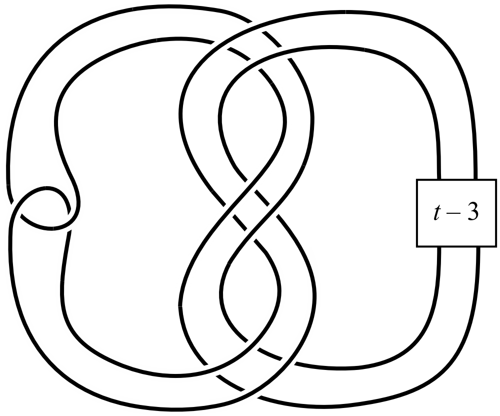

Figure 1. The t-twisted positive Whitehead double of the right-handed trefoil. The box indicates the number of full right-handed twists to insert.

(5) Knots with up to 10 crossings. See [4], [25], [18].

(6) "Most" twisted Whitehead doubles of an arbitrary knot, K. This is due to Livingston and Naik [13].

(7) Fibered knots with $\tau(K) = g(K)$. This follows from work of the first author [5].

Indeed, it was conjectured that the two invariants always coincide:

Conjecture. (Rasmussen [26]) $s(K) = 2\tau(K)$ for any knot K.

In light of the above list, the formal properties that the two invariants share, and several other striking connections between Khovanov's homology theory and Ozsváth-Szabó theory [14], [22], [27], [29], there was justified hope that the above conjecture could be true. However, we will demonstrate a counterexample:

THEOREM 1.1. *Let $D_+(T_{2,3}, 2)$ denote the 2-twisted positive Whitehead double of the right-handed trefoil knot (see Figure 1). Then $\tau(D_+(T_{2,3}, 2)) = 0$ while $s(D_+(T_{2,3}, 2)) = 2$.*

Livingston and Naik [13] calculate τ and s for all but finitely many twisted Whitehead doubles of a knot, K, in terms of the maximal Thurston-Bennequin number of K, $TB(K)$, and its reflection, $\overline{K}$. In particular, they show that $\tau(D_+(K, t)) = s(D_+(K, t))/2 = 1$ if $t \leq TB(K)$ and $\tau(D_+(K, t)) = s(D_+(K, t))/2 = 0$ if $t \geq -TB(\overline{K})$. In light of an inequality satisfied by τ and s under the operation of a crossing change, they define an invariant (which the results of this paper indicate is actually two invariants) $t_\tau(K)$ (resp. $t_s(K)$) which is the greatest integer t such that $\tau(D_+(K, t)) = 1$ (resp. $s(D_+(K, t)) = 2$). Using the techniques for the calculation above, we are able to determine $t_\tau(K)$ for the $(2, 2n + 1)$ torus knots:

THEOREM 1.2. *Let $D_+(T_{2,2n+1}, t)$ denote the t-twisted positive Whitehead double of the the $(2, 2n+1)$ torus knot. Then we have:*

$$\tau(D_+(T_{2,2n+1}, t)) = \begin{cases} 0 & \textit{for } t > 2n-1 \\ 1 & \textit{for } t \leq 2n-1. \end{cases}$$

Thus, $t_\tau(T_{2,2n+1}) = 2n - 1$. In fact, the above knots provide further counterexamples, as was shared with us by Jake Rasmussen, who used Bar-Natan's program [1] for computing Khovanov homology to calculate s for the knots in the above family which are not covered by Livingston and Naik's result. In particular,

$$\begin{aligned} s(D_+(T_{2,5}, 5)) &= s(D_+(T_{2,5}, 4)) = s(D_+(T_{2,7}, 8)) \\ &= s(D_+(T_{2,7}, 7)) = s(D_+(T_{2,7}, 6)) = 2, \end{aligned}$$

while Theorem 1.2 implies that $\tau = 0$ for these knots. It seems likely that Whitehead doubles of the $(2, 2n+1)$ torus knots provide an infinite family of counterexamples. Indeed, it would be reasonable to guess that $t_s(T_{2,2n+1}) = 3n - 1$.

In another direction, by taking connected sums of the above examples and performing a band modification of the natural Seifert surface, Livingston was able to obtain an example of a topologically slice knot, K, for which the two invariants disagree, thus proving the following:

COROLLARY 1.3. (Livingston [12]) *Let $\mathcal{C}_{ts}$ denote the subgroup of the smooth concordance group of knots consisting of those knots which are topologically slice. Then the homomorphisms*

$$s, \tau\colon \mathcal{C}_{ts} \to \mathbb{Z}$$

are independent.

We prove the above results first by calculating the knot Floer homology groups of a specific twisted Whitehead double which happens to be a $(1, 1)$ knot. A general technique for calculating the Floer homology of such knots was developed by Goda, Morifuji, and Matsuda [4] and we apply their technique here. We then use results of Eftekhary [3] for the 0-twisted Whitehead double of $T_{2,2n+1}$, together with properties of the skein exact sequence for knot Floer homology to calculate τ for the examples above. The techniques here will be refined and generalized in [6] to calculate τ and the Floer homology of an arbitrarily twisted Whitehead double of an arbitrary knot (in fact, [6] will prove that $t_\tau(K) = 2\tau(K) - 1$). We also remark that $(1, 1)$ satellite knots were classified by Morimoto and Sakuma in [15], and it was in the context of a more general study of these knots that this work arose. We hope to return to this study (see also [16]).

There is a beautiful conjectural picture due to Dunfield, Gukov, and Rasmussen, [2] of a triply graded homology theory which would unify Khovanov homology, knot Floer homology, and the various $sl(n)$ link homology theories of Khovanov and Rozansky [8]. It would be very interesting to understand this conjecture for the above examples—in particular it would be useful to calculate the $sl(n)$ link homology.

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2. Computation. In this section we compute s and τ for the examples discussed in the introduction. Before beginning, we remark that for the purposes of demonstrating a counterexample to Rasmussen's conjecture, the computation of s is much simpler. Indeed, Bar-Natan and Shumakovitch [1], [31] have independently developed computer programs which can compute the unreduced Khovanov homology of $D_+(T_{2,3}, 2)$ without difficulty. Using their programs we obtained the following Poincaré polynomial for this homology (where the t variable corresponds to the homological grading and the q variable to the Jones, or quantum, grading):

$$\begin{aligned} PKh(q,t) \;=\; & q^{-5}t^{-4} + q^{-1}t^{-3} + q^{-1}t^{-2} + qt^{-1} + q^{3}t^{-1} + 2q + q^{3} + q^{5} + 2q^{5}t \\ & + q^{5}t^{2} + q^{9}t^{2} + q^{7}t^{3} + q^{9}t^{3} + q^{7}t^{4} + q^{11}t^{4} + q^{9}t^{5} + q^{11}t^{5} + q^{13}t^{6} \\ & + q^{13}t^{7} + q^{15}t^{8} + q^{17}t^{8} + q^{19}t^{9}. \end{aligned}$$

The only homology in homological grading 0 is supported in q gradings $1, 3, 5$. It follows from the definition of s that $s(D_+(T_{2,3}, 2))$ is equal to 2 or 4. However, the fact that the genus of $D_+(T_{2,3}, 2)$ is equal to one and $|s(K)| \leq 2g_4(K)$ implies $s(D_+(T_{2,3}, 2)) = 2$.

Thus the difficulty lies in the computation of $\tau(D_+(T_{2,3}, 2))$ and this task will occupy the remainder of the paper. Our strategy will readily extend to compute τ for all twisted doubles of the $(2, 2n+1)$ torus knots. We find it interesting to note that while the computation of $s(D_+(T_{2,3}, 2))$ was easily handled by computer we see no way to approach a general computation of s for arbitrary twisted doubles of the $(2, 2n + 1)$ torus knots. Hence the results of the present paper serve to highlight the dichotomy between the quantum and gauge theoretic invariants.

2.1. Computation of τ. Recall that the knot Floer homology groups are a collection of abelian groups graded by two variables, i and t, indexed by the integers. The t variable is the homological, or Maslov grading and is analogous to the t grading for the Khovanov homology discussed above. The i variable is the Alexander, or filtration grading and is analogous to the q grading above. Its name stems from the fact that if one restricts one's attention to a single i

grading, the Euler characteristic of the knot Floer homology of K will be the i-th coefficient of the symmetrized Alexander-Conway polynomial. See [19] for more details. Throughout the rest of the paper all homology will be taken with $\mathbb{Z}/2\mathbb{Z}$-coefficients, and we will denote this coefficient field by $\mathbb{F}$. Hence all homology groups will be vector spaces and this will simplify some of the discussion. We use the notation $\mathbb{F}^k_{(t)}$ to denote a $\mathbb{Z}/2\mathbb{Z}$ vector space of dimension k, supported in homological grading t. The symbol $\widehat{HFK}(K,i)$ denotes the direct sum over all homological gradings of the knot Floer homology groups supported in Alexander grading i.

In the following proposition we calculate the knot Floer homology groups of $D_+(T_{2,3},6)$, the 6-twisted positive Whitehead double of the right-handed trefoil. For notational simplicity we use $D(t)$ to mean the t-twisted positive Whitehead double of the right-handed trefoil.

Proposition 2.1.

$$\widehat{HFK}(D(6),i) \cong \begin{cases} \mathbb{F}^4_{(1)} \oplus \mathbb{F}^2_{(-1)} & \textit{for } i = 1 \\ \mathbb{F}^9_{(0)} \oplus \mathbb{F}^4_{(-2)} & \textit{for } i = 0 \\ \mathbb{F}^4_{(-1)} \oplus \mathbb{F}^2_{(-3)} & \textit{for } i = -1 \end{cases}$$

Remark. Note that $\tau(D(6)) = 0$. There is simply no homology in grading 0 supported in filtration grading 1 or -1.

Proof. We first apply the technique developed in [4] for obtaining a genus one doubly-pointed Heegaard diagram from a $(1,1)$ presentation to the knot at hand, $D(6)$. This is illustrated in Figure 2. Following the technique which Ozsváth and Szabó introduced in Section 6 of [19] (and which was further developed by [4]), we lift this genus one diagram to the universal cover, Figure 3, and compute the boundary map:

$$\begin{aligned}
\partial[x_1,i,i] &= 0 \\
\partial[x_2,i,i+1] &= [x_1,i,i]+[x_5,i-1,i-1] \\
\partial[x_3,i,i] &= [x_2,i-1,i]+[x_4,i,i-1] \\
\partial[x_4,i,i-1] &= [x_1,i-1,i-1]+[x_5,i-2,i-2] \\
\partial[x_5,i,i] &= 0 \\
\partial[x_6,i,i+1] &= [x_5,i,i]+[x_9,i,i] \\
\partial[x_7,i,i] &= [x_6,i-1,i]+[x_8,i,i-1] \\
\partial[x_8,i,i-1] &= [x_5,i-1,i-1]+[x_9,i-1,i-1] \\
\partial[x_9,i,i] &= 0 \\
\partial[x_{10},i,i+1] &= [x_9,i,i]+[x_{13},i,i]
\end{aligned}$$

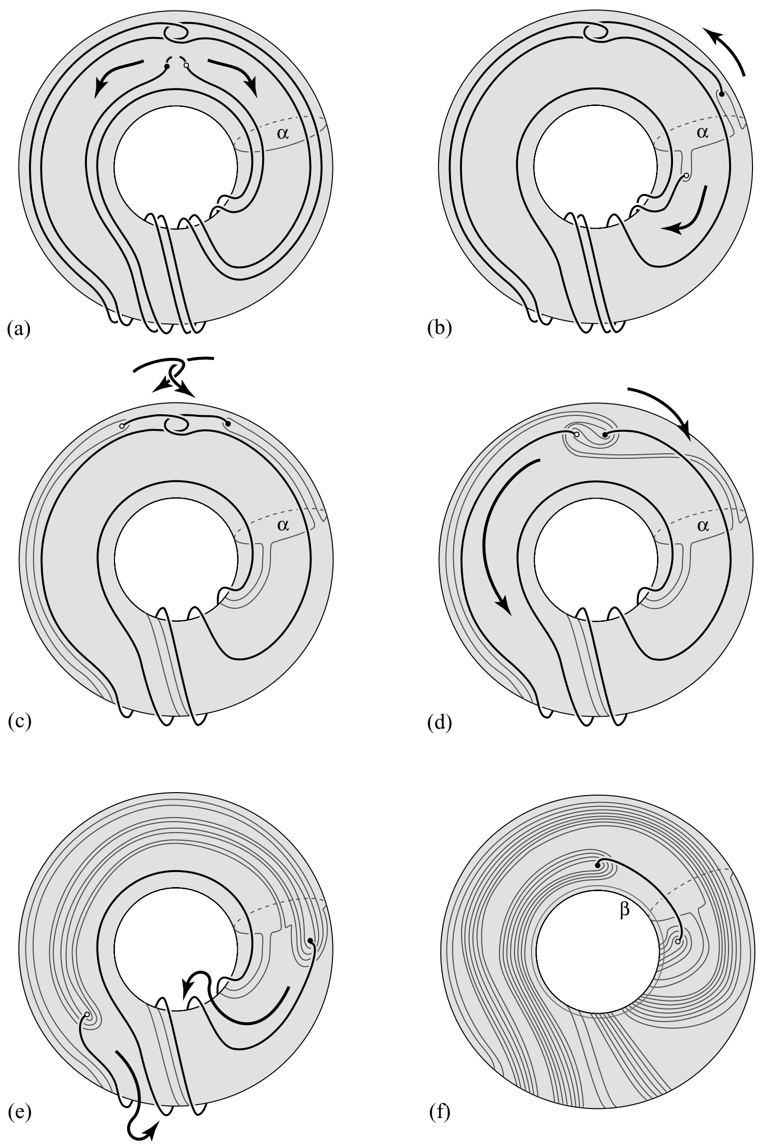

Figure 2. Construction of a doubly-pointed Heegaard diagram (f) for the Whitehead double $D(6)$ of the trefoil, from a $(1,1)$ presentation (a).

$$\begin{aligned}
\partial[x_{11},i,i] &= [x_{10},i-1,i]+[x_{12},i,i-1]\\
\partial[x_{12},i,i-1] &= [x_{9},i-1,i-1]+[x_{13},i-1,i-1]\\
\partial[x_{13},i,i] &= 0\\
\partial[x_{14},i,i+1] &= [x_{13},i,i]+[x_{17},i,i]\\
\partial[x_{15},i,i] &= [x_{14},i-1,i]+[x_{16},i,i-1]\\
\partial[x_{16},i,i-1] &= [x_{13},i-1,i-1]+[x_{17},i-1,i-1]
\end{aligned}$$

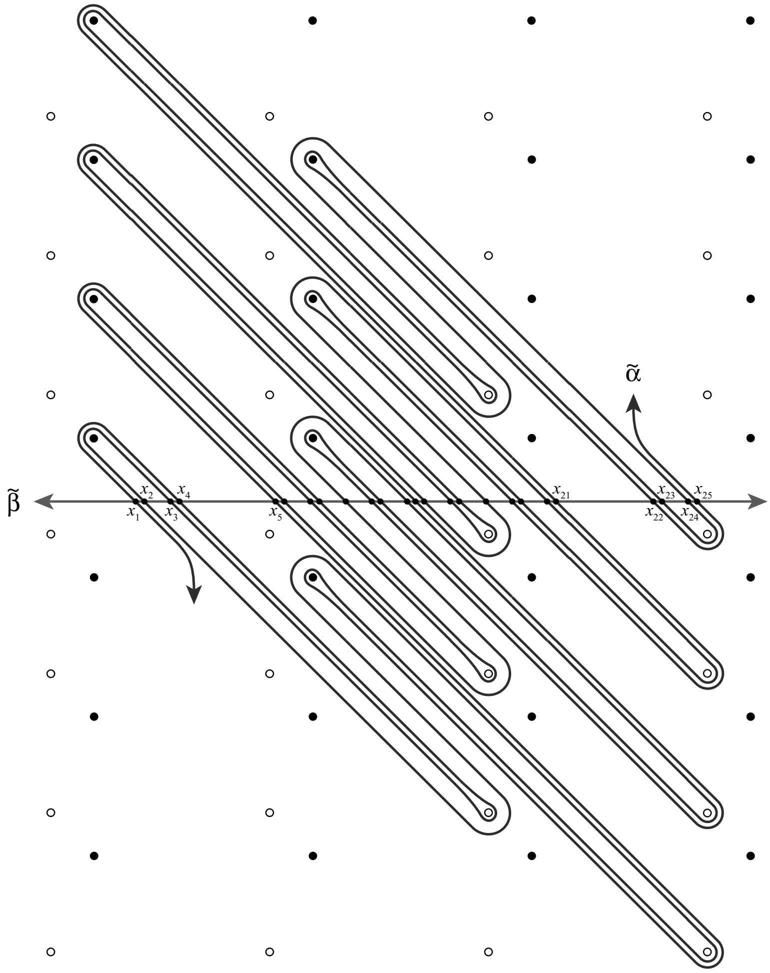

Figure 3. The Heegaard diagram of the previous figure, lifted to the universal cover of the torus. We have chosen a particular lift of α and β, as indicated. The open circles denote lifts of the basepoint z while the black circles denote lifts of w.

$$
\begin{aligned}
\partial[x_{17}, i, i] &= 0 \\
\partial[x_{18}, i, i+1] &= [x_{17}, i, i] + [x_{21}, i, i] \\
\partial[x_{19}, i, i] &= [x_{18}, i-1, i] + [x_{20}, i, i-1] \\
\partial[x_{20}, i, i-1] &= [x_{17}, i-1, i-1] + [x_{21}, i-1, i-1] \\
\partial[x_{21}, i, i] &= 0 \\
\partial[x_{22}, i, i+1] &= [x_{25}, i, i] + [x_{21}, i-1, i-1] \\
\partial[x_{23}, i, i] &= [x_{22}, i-1, i] + [x_{24}, i, i-1] \\
\partial[x_{24}, i, i-1] &= [x_{25}, i-1, i-1] + [x_{21}, i-2, i-2] \\
\partial[x_{25}, i, i] &= 0.
\end{aligned}
$$

Table 1.

	-3	-2	-1	0	1
1			x_2, x_{22}		$x_6, x_{10}, x_{14}, x_{18}$
0		x_1, x_3, x_{23}, x_{25}		$x_5, x_7, x_9, x_{11}, x_{13}, x_{15}, x_{17}, x_{19}, x_{21}$	
-1	x_4, x_{24}		$x_8, x_{12}, x_{16}, x_{20}$		

Using our knowledge of the differential, it is easy to separate the generators of the chain complex into their respective filtration and homological gradings. In Table 1 the vertical (horizontal) direction indicates the filtration (homological) grading. The proposition follows immediately. □

Next we recall the following result of Eftekhary:

THEOREM 2.2. (Eftekhary [3])

$$\widehat{HFK}(D(0), 1) \cong \mathbb{F}^2_{(t)} \oplus \mathbb{F}^2_{(t-1)},$$

where the subscript (t) indicates that the homological grading is known only as a relative $\mathbb{Z}$-grading.

By performing six successive crossing changes to the twisting region of the knot diagram shown in Figure 1, we can change $D(6)$ into $D(0)$. Each of these operations changes a negative crossing to a positive crossing. Recall that Theorem 10.2 of [19] (see also [25]) asserts that associated to a crossing change there are skein exact sequences for knot Floer homology (for each i):

$$\cdots \longrightarrow \widehat{HFK}(K_-, i) \xrightarrow{f_1} \widehat{HFK}(S^1 \times S^2, \kappa(K_0), i) \xrightarrow{f_2} \widehat{HFK}(K_+, i) \xrightarrow{f_3} \cdots,$$

where the maps f_1 and f_2 lower homological grading by one-half and f_3 is non-increasing in the homological grading. Here K_- is the knot with negative crossing, K_+ is the knot with positive crossing, and K_0 is the two-component link obtained by resolving the crossing. More precisely, Section 2 of [19] describes a well-defined way to associate a knot $(S^1 \times S^2, \kappa(L))$ to a two-component link (S^3, L) and $(S^1 \times S^2, \kappa(K_0))$ is this "knotification" of the link obtained from resolving the crossing. We also note that Ozsváth and Szabó define an absolute $\mathbb{Z}/2\mathbb{Z}$ homological grading on the groups in the above sequence which is simply the parity of the homological grading. With respect to the $\mathbb{Z}/2\mathbb{Z}$ grading, the maps f_1 and f_3 are grading-preserving, while f_2 is grading-reversing. Note that the underlying three-manifold in the middle term is $S^1 \times S^2$, which has Floer homology graded by half integers, $\{-\frac{1}{2} + n\}_{n\in\mathbb{Z}}$. The absolute $\mathbb{Z}/2\mathbb{Z}$ grading for $\widehat{HFK}(S^1 \times$

$S^2, \kappa(K_0))$ is determined by the convention that $\frac{1}{2} + 2n$ is odd while $-\frac{1}{2} + 2n$ is even.

Thus the remaining step in our computation of $\tau(D(2))$ will be to study the skein exact sequences associated to the six aforementioned crossing changes. These exact sequences relate the Floer homology groups of $D(t)$, $D(t-1)$ and the two-component link obtained from the oriented resolution of the crossing which we change. For each t, this link is the positive Hopf link, which we denote by H. The Floer homology of the knotification of H is given by:

PROPOSITION 2.3.

$$\widehat{HFK}(S^1 \times S^2, \kappa(H), i) \cong \begin{cases} \mathbb{F}_{(\frac{1}{2})} & \text{if } i = 1 \\ \mathbb{F}^2_{(-\frac{1}{2})} & \text{if } i = 0 \\ \mathbb{F}_{(-\frac{3}{2})} & \text{if } i = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This was originally proved in Proposition 9.2 of [19], but also follows easily from [17], whose main theorem determines the Floer homology of alternating links in terms of their Alexander polynomial and signature. □

In light of the above, we see that the exact sequence for the top filtration level takes the following form:

$$\cdots \longrightarrow \widehat{HFK}(D(t), 1) \xrightarrow{f_1} \mathbb{F}_{(\frac{1}{2})} \xrightarrow{f_2} \widehat{HFK}(D(t-1), 1) \xrightarrow{f_3} \cdots$$

It follows at once that there are two options for each skein sequence:

(1) $f_2 = 0, f_1 \neq 0$

(2) $f_2 \neq 0, f_1 = 0$

We make the following claim:

PROPOSITION 2.4. *In the exact sequence above relating $D(t)$, $D(t-1)$ and H, $f_2 \neq 0$ if and only if $\tau(D(t-1)) = 1$. Otherwise $\tau(D(t-1)) = 0$.*

Remark. It follows independently from the work of Livingston and Naik [13] that $\tau(D(t))$ is equal to 0 or 1.

Proof. The proposition will follow from the fact that f_2 is the lowest order term in a filtered chain map, $\tilde{f}_2$, between chain complexes which are chain homotopy equivalent to $\widehat{CF}(S^1 \times S^2)$ and $\widehat{CF}(S^3)$, respectively.

To begin, note that the Floer homology groups for H (resp. $D(t-1)$) are endowed with an induced differential which gives them the structure of a filtered chain complex. Moreover, this differential strictly lowers the filtration index. In the case of H, the homology of this filtered chain complex is $\widehat{HF}(S^1 \times S^2) \cong$

$\mathbb{F}_{(-\frac{1}{2})} \oplus \mathbb{F}_{(\frac{1}{2})}$. In the case of $D(t-1)$, the homology is $\widehat{HF}(S^3) \cong \mathbb{F}_{(0)}$. The filtration on the knot Floer homology of $D(t-1)$ induces a filtration on $\widehat{HF}(S^3)$ in the standard way, i.e. the filtration level of any cycle, $z = \Sigma n_x x$, is by definition the maximum filtration level of any chain x which comprises z. Now $\tau(D(t-1))$ is defined to be the minimum filtration grading of any cycle $z \in \widehat{HFK}(D(t-1))$ which is homologous to a generator of $\widehat{HF}(S^3)$.

It follows from the proof of the skein sequence (Theorem 8.2 of [19]) that there is a filtered chain map

$$\tilde{f_2}\colon\ \widehat{HFK}(S^1 \times S^2, \kappa(H)) \to \widehat{HFK}(D(t-1)).$$

The map $\tilde{f_2}$ decomposes as a sum of homogeneous pieces, each of which lower the filtration by some fixed integer. The map in the skein sequence is the part of $\tilde{f_2}$ which preserves the filtration.

From Proposition 2.3 we see that a chain generating $\widehat{HFK}(S^1 \times S^2, \kappa(H), 1) \cong \mathbb{F}_{(\frac{1}{2})}$ is a cycle under the induced differential, and hence the above discussion implies that $\tilde{f_2}$ maps this chain to a cycle, $z \in \widehat{HFK}(D(t-1))$. Now if f_2 is nontrivial, z contains nontrivial chains with filtration index 1. The definition of τ, together with the fact that $\widehat{HFK}(D(t-1), i) \cong 0$ for $i > 1$ implies $\tau(D(t-1)) = 1$.

Now on the level of homology, $\tilde{f_2}$ induces a map:

$$\widehat{HF}(S^1 \times S^2) \cong \mathbb{F}_{(-\frac{1}{2})} \oplus \mathbb{F}_{(\frac{1}{2})} \xrightarrow{(\tilde{f_2})_*} \widehat{HF}(S^3) \cong \mathbb{F}_{(0)}$$

which sends the space supported in grading one-half to the generator. If $\tau(D(t-1)) = 1$, the cycle generating $\widehat{HF}(S^3)$ contains nontrivial chains in filtration level 1. It follows that f_2—the part of $\tilde{f_2}$ which preserves the filtration—is nontrivial.

Finally, if $\tau(D(t)) = -1$, a similar analysis shows that $\tilde{f_1}$ restricted to $\widehat{HFK}(D(t), -1)$ would raise the filtration grading, contradicting the fact that this map respects the filtration. □

The above proposition shows that the map f_2 in the skein sequence controls the behavior of $\tau(D(t-1))$. We determine when f_2 is nontrivial in the six applications of the sequence:

LEMMA 2.5. *The map* f_2: $\widehat{HFK}(H, 1) \to \widehat{HFK}(D(t-1), 1)$ *is trivial for* $t = 6, 5, 4, 3$ *and nontrivial for* $t = 2, 1$.

Theorem 1.1 will follow immediately from the above lemma and Proposition 2.4. Indeed, it follows easily from the proof that $\tau(D(t)) = 0$ if $t > 1$ and $\tau(D(t)) = 1$ if $t \leq 1$.

Proof. We study the function

$$e(t) = \mathrm{rk}_{even}\widehat{HFK}(D(t), 1)$$

which measures the rank of the Floer homology in top filtration level supported in even homological grading.

We claim that if f_2 is nontrivial then $e(t-1) = e(t) + 1$ and if f_2 is trivial then $e(t-1) = e(t)$. This follows from the form of the skein sequence at hand, together with the knowledge that f_1 and f_3 preserve the $\mathbb{Z}/2\mathbb{Z}$-grading while f_2 reverses it. From Proposition 2.1 and Theorem 2.2 we have $e(6) = 0$ and $e(0) = 2$. Thus our claim shows that among the six applications of the skein sequence, f_2 is nontrivial exactly twice.

Next, recall that τ (and s) satisfy the following inequality under the operation of changing a crossing in a given knot diagram (see [11] or [18] for a proof):

$$\tau(K_+) - 1 \le \tau(K_-) \le \tau(K_+),$$

where K_+ (resp. K_-) denotes the diagram with the positive (resp. negative) crossing. Now each application of the skein sequence arose from changing a single negative crossing to a positive crossing. Hence the above inequality becomes (for $k > 0$):

$$\tau(D(t-k)) - k \le \tau(D(t)) \le \tau(D(t-k)).$$

If f_2 were nontrivial for some t and trivial for $t-k$, then Proposition 2.4 would imply $\tau(D(t-1)) = 1$ and $\tau(D(t-k-1) = 0$ violating the inequality. Thus f_2 is trivial for $t = 6, 5, 4, 3$ as stated, and nontrivial for $t = 2, 1$. □

2.2. Twisted Whitehead doubles of $(2, 2n{+}1)$ torus knots. Let $D_+(T_{2,2n+1}, t)$ denote the t-twisted positive Whitehead double of the right-handed $(2, 2n{+}1)$ torus knot. Results of [15] indicate that the $D_+(T_{2,2n+1}, 4n{+}2)$ is a $(1, 1)$ knot, and indeed we can repeat the calculation of Proposition 2.1 to yield:

PROPOSITION 2.6.

$$\widehat{HFK}(D_+(T_{2,2n+1}, 4n+2), i) \cong \begin{cases} \mathbb{F}^{2n+2}_{(1)} \oplus \mathbb{F}^2_{(-1)} \oplus \mathbb{F}^2_{(-3)} \cdots \oplus \mathbb{F}^2_{(-2n+1)} & \text{for } i = 1 \\ \mathbb{F}^{4n+5}_{(0)} \oplus \mathbb{F}^4_{(-2)} \oplus \mathbb{F}^4_{(-4)} \cdots \oplus \mathbb{F}^4_{(-2n)} & \text{for } i = 0 \\ \mathbb{F}^{2n+2}_{(-1)} \oplus \mathbb{F}^2_{(-3)} \oplus \mathbb{F}^2_{(-5)} \cdots \oplus \mathbb{F}^2_{(-2n-1)} & \text{for } i = -1. \end{cases}$$

In addition, Eftekhary's [3] results in this case yield:

THEOREM 2.7.

$$\widehat{HFK}(D_+(T_{2,2n+1}, 0), 1) \cong \mathbb{F}^{2n}_{(t)} \oplus \mathbb{F}^2_{(t-1)} \oplus \mathbb{F}^2_{(t-3)} \ldots \oplus \mathbb{F}^2_{(t-2n+1)}.$$

The technique for computing τ in the case of the trefoil can now be applied to yield Theorem 1.2.

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