Superdense Coding
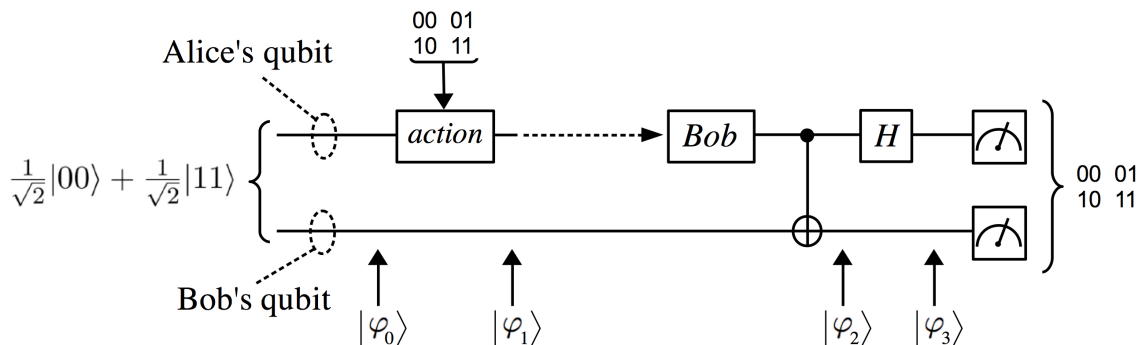

One day Alice and Bob go down to their local Walmart store and purchase a brand new pair of entangled qubits. Each qubit comes factory-sealed in its own attractive box designed to prevent interactions with the surrounding environment. One qubit of the pair is for Alice, and the other is for Bob. The pair they chose was created in the following entangled state:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Alice puts her qubit aside for now, and Bob returns home to Mars, taking his qubit with him. Neither Bob nor Alice has — as yet — measured their respective qubits in any way, or even opened the packaging. Sometime later, the need arises for Alice to communicate *two classical bits* of information to Bob on Mars. To accomplish this, she decides to use her own qubit (which is still entangled with Bob's). She carefully unpacks and removes her qubit from its box, without measuring it, and then applies one of the following actions to it, depending on which two classical bits she wants to send to Bob:

Bits to send	Alice's action	Resulting entangled qubit state	
00	Do nothing	$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$	$= \left[\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right]^T$
01	Apply X	$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$	$= \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right]^T$
10	Apply Z	$\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$	$= \left[\frac{1}{\sqrt{2}}, 0, 0, \frac{-1}{\sqrt{2}}\right]^T$
11	Apply ZX	$\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$	$= \left[0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right]^T$

The resulting entangled states are called *Bell states*, and they form an orthonormal basis in the 2-qubit state space. Alice now sends her qubit to Bob. After receiving Alice's qubit, Bob applies a *CNOT* gate to both qubits, and then an H gate to the top qubit (the one he received from Alice). Finally, he measures both qubits, which yield the two classical bits. In this way, Alice has managed to transmit two bits of information using a single (entangled) qubit.