Variance of an observable

If we prepare a quantum system in the state $|\psi\rangle$, and then observe the value of Ω, we will get some real number as a result (one of Ω's eigenvalues). If we repeatedly do this (meaning that we prepare the system again in state $|\psi\rangle$ and then observe Ω again), we will get a series of real numbers. Over time, with enough repeated observations of Ω in state $|\psi\rangle$, we will obtain an average value that approximates $\langle\psi|\,\Omega\,|\psi\rangle$. The actual real numbers obtained will generally deviate from $\langle\psi|\,\Omega\,|\psi\rangle$, sometimes being bigger and sometimes smaller. In addition to the expected mean value of Ω, we can calculate the expected amount of *variation around the mean value*, also known as the *variance* of Ω. Since the difference between an observed value and the mean can be positive or negative (over or under the mean), we will use the *square* of the difference in computing the variance, rather than the raw difference itself; that way, we won't have to worry about negative signs.

We will modify Ω slightly, in such a way as to produce the *differences* between the observed values and the expected mean $\mu = \langle\psi|\,\Omega\,|\psi\rangle = \langle\Omega\rangle_\psi$, rather than the actual observed values themselves. As a concrete example, suppose that $\langle\psi|\,\Omega\,|\psi\rangle = 3.3$, meaning that repeatedly measuring Ω in state $|\psi\rangle$ gives a sequence of values that average out to 3.3 in the long run, such as: 2.5, 4.0, 3.4, 4.9, 1.7. We modify Ω by transforming it with a new operator Δ_ψ into $\Delta_\psi(\Omega)$, which also depends on $|\psi\rangle$. When we repeatedly measure $\Delta_\psi(\Omega)$ in state $|\psi\rangle$, we get a sequence of "demeaned" values with the same relative distribution as before, but whose mean is 0. For example: -0.8, 0.7, 0.1, 1.6, -1.6.

The $\Delta_\psi(\Omega)$ operator simply subtracts the mean value $\mu = \langle\Omega\rangle_\psi$ from the diagonal entries of Ω:

$$\Delta_\psi(\Omega) = \Omega - \langle\Omega\rangle_\psi\, I$$

Suppose $\Omega = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ with expected value $\mu = \langle\Omega\rangle_\psi$ in state $|\psi\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\Delta_\psi(\Omega) = \Omega - \mu I = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} - \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} \lambda_1 - \mu & 0 \\ 0 & \lambda_2 - \mu \end{bmatrix}$$

Let's compare the actions of Ω and $\Delta_\psi(\Omega)$ on $|\psi\rangle$:

$$\Omega|\psi\rangle = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{bmatrix}$$

$$\Delta_\psi(\Omega)|\psi\rangle = \begin{bmatrix} \lambda_1 - \mu & 0 \\ 0 & \lambda_2 - \mu \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (\lambda_1 - \mu)c_1 \\ (\lambda_2 - \mu)c_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 - \mu c_1 \\ \lambda_2 c_2 - \mu c_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{bmatrix} - \begin{bmatrix} \mu c_1 \\ \mu c_2 \end{bmatrix} = \Omega|\psi\rangle - \mu|\psi\rangle$$

What is the expected value of $\Delta_\psi(\Omega)$ itself in the state $|\psi\rangle$? It is just $\langle\psi|\,\Delta_\psi(\Omega)\,|\psi\rangle$, which is:

$$\langle\psi|\Big(\Omega|\psi\rangle - \mu|\psi\rangle\Big) = \langle\psi|\Omega|\psi\rangle - \langle\psi|\mu|\psi\rangle = \mu - \mu\langle\psi\,|\,\psi\rangle = 0 \qquad \text{since } \langle\psi\,|\,\psi\rangle = 1$$

We can now consider the amount of variation in the values of $\Delta_\psi(\Omega)$. That is, how wide on average is their "spread" or deviation from the mean value of 0? The expected value of this deviation—*squared*, so that we can ignore whether deviations are negative or positive—is the *variance* of Ω in state $|\psi\rangle$, denoted $\mathrm{Var}_\psi(\Omega) = \langle\Delta_\psi(\Omega) \star \Delta_\psi(\Omega)\rangle_\psi$. Like the mean, it is just a real number.

Variance example

Consider the observable property Ω, with the following eigenvalues and orthonormal eigenbasis:

$$\Omega = \begin{bmatrix} 3 & 1+i \\ 1-i & 2 \end{bmatrix} \qquad \lambda_1 = 4, \quad |e_1\rangle = \begin{bmatrix} \frac{1+i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \qquad \lambda_2 = 1, \quad |e_2\rangle = \begin{bmatrix} \frac{-1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

- When we measure Ω in the laboratory, the outcome of the experiment on our measuring device will always be either 4 or 1.
- If we prepare the system in state $|up\rangle$ and then measure Ω once, what is the probability of obtaining 4? What is the probability of obtaining 1?

$$\langle e_1 \,|\, up\rangle = \begin{bmatrix} \left(\frac{1+i}{\sqrt{3}}\right)^* & \left(\frac{1}{\sqrt{3}}\right)^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1-i}{\sqrt{3}} \qquad \text{So probability of obtaining } 4 = \left|\frac{1-i}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

$$\langle e_2 \,|\, up\rangle = \begin{bmatrix} \left(\frac{-1-i}{\sqrt{6}}\right)^* & \left(\frac{2}{\sqrt{6}}\right)^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{-1+i}{\sqrt{6}} \qquad \text{So probability of obtaining } 1 = \left|\frac{-1+i}{\sqrt{6}}\right|^2 = \frac{1}{3}$$

- If we perform this experiment many times by repeatedly preparing the system in state $|up\rangle$ and then measuring Ω, what will be the *average* of the values obtained over the long run? This is the *expected value* of Ω in state $|up\rangle$, also sometimes written as $\langle\Omega\rangle_{up}$.

  Weighted average of eigenvalues $= \frac{2}{3} \cdot \lambda_1 + \frac{1}{3} \cdot \lambda_2 = \frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 1 = 3$

$$\langle up|\,\Omega\,|up\rangle = \begin{bmatrix} 1^* & 0^* \end{bmatrix} \begin{bmatrix} 3 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1^* & 0^* \end{bmatrix} \begin{bmatrix} 3 \\ 1-i \end{bmatrix} = 3$$

- If we prepare the system in state $|e_1\rangle$ and measure Ω, what are the probabilities of obtaining 4 or 1? What is the expected value?

  $\langle e_1 \,|\, e_1\rangle = 1$ So probability of obtaining $4 = |1|^2 = 1$

  $\langle e_2 \,|\, e_1\rangle = 0$ So probability of obtaining $1 = |0|^2 = 0$

$$\langle e_1|\,\Omega\,|e_1\rangle = \begin{bmatrix} \left(\frac{1+i}{\sqrt{3}}\right)^* & \left(\frac{1}{\sqrt{3}}\right)^* \end{bmatrix} \begin{bmatrix} 3 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} \frac{1+i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{4(1+i)}{\sqrt{3}} \\ \frac{4}{\sqrt{3}} \end{bmatrix} = \frac{8}{3} + \frac{4}{3} = 4$$

  The outcome will always be 4 (that is, λ_1) when we measure Ω in the eigenstate $|e_1\rangle$.

- We can quantify the amount of "fuzziness" or uncertainty of a measurement by calculating the *variance* of Ω when measured in a state $|\psi\rangle$, written as $\text{Var}_\psi(\Omega)$. If we repeatedly measure Ω in state $|up\rangle$, the amount of expected variation in the measurements obtained over the long run will be $\text{Var}_{up}(\Omega)$. Likewise, the amount of expected variation when we measure Ω in state $|e_1\rangle$ will be $\text{Var}_{e_1}(\Omega)$.

To calculate $\text{Var}_\psi(\Omega)$, we first define a new observable $\Delta_\psi(\Omega)$ based on Ω and the expected value μ of Ω in state $|\psi\rangle$. This new observable is called the *mean-adjusted* or *demeaned* version of Ω in state $|\psi\rangle$:

- Expected value $\mu = \langle\psi|\,\Omega\,|\psi\rangle$
- $\Delta_\psi(\Omega) = \Omega - \mu I = \Omega - \begin{bmatrix}\mu & 0\\ 0 & \mu\end{bmatrix} = \begin{bmatrix}3-\mu & 1+i\\ 1-i & 2-\mu\end{bmatrix}$

- Expected value $\langle up|\,\Omega\,|up\rangle = 3$
- $\Delta_{up}(\Omega) = \begin{bmatrix}3 & 1+i\\ 1-i & 2\end{bmatrix} - \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix} = \begin{bmatrix}0 & 1+i\\ 1-i & -1\end{bmatrix}$

- Expected value $\langle e_1|\,\Omega\,|e_1\rangle = 4$
- $\Delta_{e_1}(\Omega) = \begin{bmatrix}3 & 1+i\\ 1-i & 2\end{bmatrix} - \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} = \begin{bmatrix}-1 & 1+i\\ 1-i & -2\end{bmatrix}$

We then square $\Delta_\psi(\Omega)$ to produce another new observable: $\Delta_\psi(\Omega) \star \Delta_\psi(\Omega)$:

- $\Delta_{up}(\Omega) \star \Delta_{up}(\Omega) = \begin{bmatrix}0 & 1+i\\ 1-i & -1\end{bmatrix}\begin{bmatrix}0 & 1+i\\ 1-i & -1\end{bmatrix} = \begin{bmatrix}2 & -1-i\\ -1+i & 3\end{bmatrix}$
- $\Delta_{e_1}(\Omega) \star \Delta_{e_1}(\Omega) = \begin{bmatrix}-1 & 1+i\\ 1-i & -2\end{bmatrix}\begin{bmatrix}-1 & 1+i\\ 1-i & -2\end{bmatrix} = \begin{bmatrix}3 & -3-3i\\ -3+3i & 6\end{bmatrix}$

The variance is the *expected value* in state $|\psi\rangle$ of this new observable:

- $\text{Var}_\psi(\Omega) = \langle\psi|\,\Delta_\psi(\Omega) \star \Delta_\psi(\Omega)\,|\psi\rangle$
- $\text{Var}_{up}(\Omega) = \langle up|\,\Delta_{up}(\Omega) \star \Delta_{up}(\Omega)\,|up\rangle$

  $$= \begin{bmatrix}1^* & 0^*\end{bmatrix}\begin{bmatrix}2 & -1-i\\ -1+i & 3\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$
  $$= 2$$

- $\text{Var}_{e_1}(\Omega) = \langle e_1|\,\Delta_{e_1}(\Omega) \star \Delta_{e_1}(\Omega)\,|e_1\rangle$

  $$= \begin{bmatrix}\left(\frac{1+i}{\sqrt{3}}\right)^* & \left(\frac{1}{\sqrt{3}}\right)^*\end{bmatrix}\begin{bmatrix}3 & -3-3i\\ -3+3i & 6\end{bmatrix}\begin{bmatrix}\frac{1+i}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\left(\frac{1+i}{\sqrt{3}}\right)^* & \left(\frac{1}{\sqrt{3}}\right)^*\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix}$$
  $$= 0$$

- Thus the uncertainty of the observable Ω is non-zero when measured in state $|up\rangle$, but there is no uncertainty at all when Ω is measured in its eigenstate $|e_1\rangle$.