

How to Compute 2^{10}

$$1 \times 2 = 2$$

$$2 \times 2 = 4$$

$$4 \times 2 = 8$$

$$8 \times 2 = 16$$

$$16 \times 2 = 32$$

$$32 \times 2 = 64$$

$$64 \times 2 = 128$$

$$128 \times 2 = 256$$

$$256 \times 2 = 512$$

$$512 \times 2 = 1024$$



Standard Exponentiation Algorithm

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ b \times b^{n-1} & \text{if } n > 0 \end{cases}$$

$$\begin{aligned} 2^3 &= 2 \times 2^2 \\ &= 2 \times 2 \times 2^1 \\ &= 2 \times 2 \times 2 \times 2^0 \\ &= 2 \times 2 \times 2 \times 1 \end{aligned}$$

3 multiplications

$$2^{1000}$$

1000 multiplications

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$$16 \times 2 = 32$$

$$32^2 = 1024 \checkmark$$

Fast Exponentiation Algorithm

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{n}{2}})^2 & \text{if } n > 0 \text{ and } n \text{ is even} \\ b \times b^{n-1} & \text{if } n > 0 \text{ and } n \text{ is odd} \end{cases}$$

Best case example: b^{32}

n	result
32	$(b^{16})^2$
16	$((b^8)^2)^2$
8	$((b^4)^2)^2)^2$
4	$((b^2)^2)^2)^2)^2$
2	$((b^1)^2)^2)^2)^2)^2$
1	$((b \times b^0)^2)^2)^2)^2)^2$
0	$((b \times 1)^2)^2)^2)^2)^2$

6 multiplications

about $\log_2(n)$ multiplications in the best case

exact # of multiplications: $\log_2(n) + 1$

Worst case example: b^{31}

n	result
31	$b \times b^{30}$
30	$b \times (b^{15})^2$
15	$b \times (b \times b^{14})^2$
14	$b \times (b \times (b^7)^2)^2$
7	$b \times (b \times (b \times b^6)^2)^2$
6	$b \times (b \times (b \times (b^3)^2)^2)^2$
3	$b \times (b \times (b \times (b \times b^2)^2)^2)^2$
2	$b \times (b \times (b \times (b \times (b^1)^2)^2)^2)^2$
1	$b \times (b \times (b \times (b \times (b \times b^0)^2)^2)^2)^2$
0	$b \times (b \times (b \times (b \times (b \times 1)^2)^2)^2)^2$

9 multiplications

about $2 \log_2(n)$ multiplications in the worst case

exact # of multiplications: $2 \lfloor \log_2(n) \rfloor + 1$