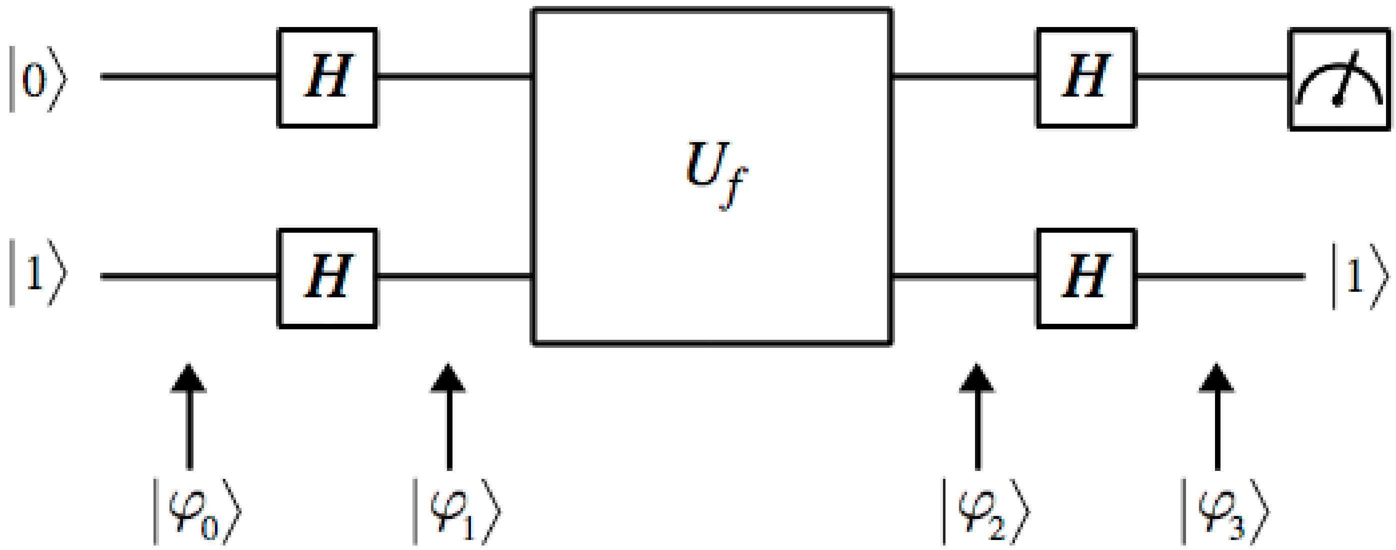
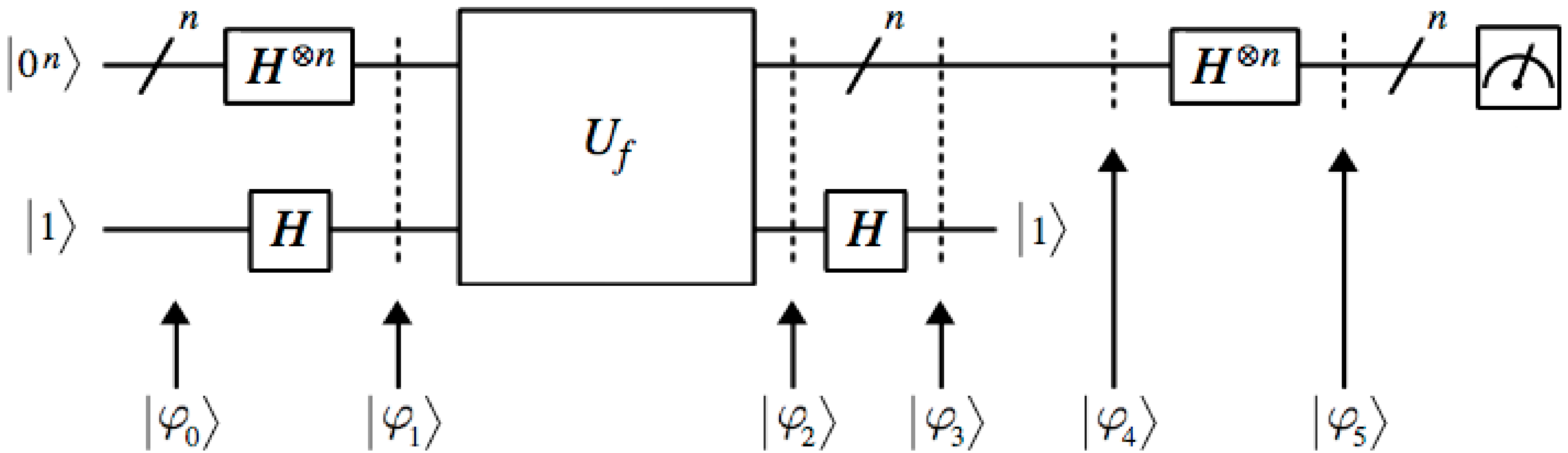
Deutsch's Algorithm*

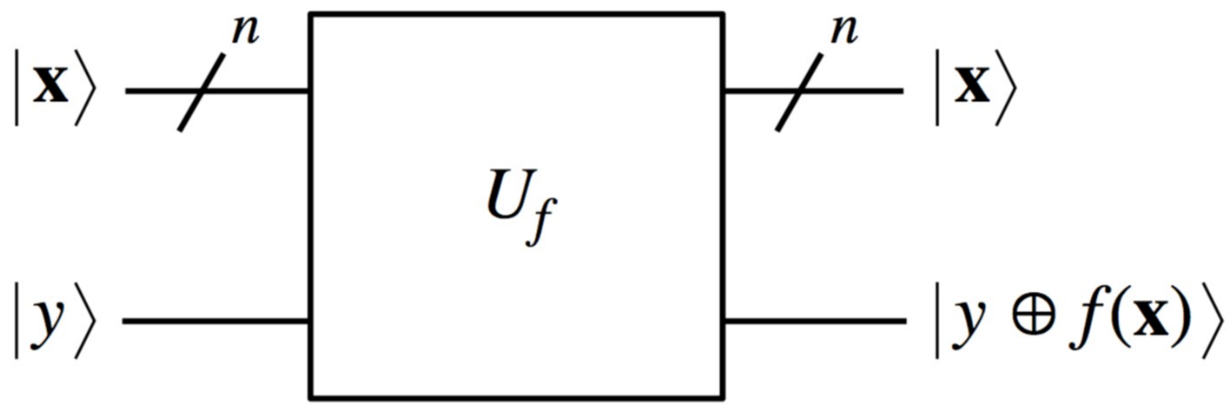


* includes Marshall's variation

Deutsch-Jozsa Algorithm*



* includes Marshall's variation



When $|y\rangle$ is the basis state $|0\rangle$ or $|1\rangle$, applying U_f gives:

- $U_f\Big(|\mathbf{x}\rangle \otimes |0\rangle\Big) = |\mathbf{x}\rangle \otimes |0 \oplus f(\mathbf{x})\rangle = |\mathbf{x}\rangle \otimes |f(\mathbf{x})\rangle$
- $U_f\Big(|\mathbf{x}\rangle \otimes |1\rangle\Big) = |\mathbf{x}\rangle \otimes |1 \oplus f(\mathbf{x})\rangle = |\mathbf{x}\rangle \otimes |\overline{f(\mathbf{x})}\rangle$

When $|y\rangle$ is the superposition state $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, applying U_f gives:

$$U_f \left(|\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right) \right)$$

$$= U_f \left(\underbrace{|\mathbf{x}\rangle \otimes}_{} \frac{1}{\sqrt{2}}|0\rangle - \underbrace{|\mathbf{x}\rangle \otimes}_{} \frac{1}{\sqrt{2}}|1\rangle \right)$$

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$$U_f\left(|\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\right)$$

$$= U_f\left(|\mathbf{x}\rangle \otimes \frac{1}{\sqrt{2}}|0\rangle - |\mathbf{x}\rangle \otimes \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= U_f\left(\frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle - \frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right)$$

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$$= U_f\left(\frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle - \frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right)$$

$$= U_f\left(\frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle\right) - U_f\left(\frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right) \qquad \text{since } U_f \text{ is a linear operator}$$

When $|y\rangle$ is the superposition state $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, applying U_f gives:

$$U_f\left(|\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\right)$$

$$= U_f\left(|\mathbf{x}\rangle \otimes \frac{1}{\sqrt{2}}|0\rangle - |\mathbf{x}\rangle \otimes \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= U_f\left(\frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle - \frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right)$$

$$= U_f\left(\frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle\right) - U_f\left(\frac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right) \qquad \text{since } U_f \text{ is a linear operator}$$

$$= \frac{1}{\sqrt{2}}U_f\left(|\mathbf{x}\rangle \otimes |0\rangle\right) - \frac{1}{\sqrt{2}}U_f\left(|\mathbf{x}\rangle \otimes |1\rangle\right) \qquad \text{since } U_f \text{ is a linear operator}$$

When $|y\rangle$ is the superposition state $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, applying U_f gives:

$$U_f\left(|\mathbf{x}\rangle \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\right)$$

$$= U_f\left(|\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|0\rangle - |\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$= U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle - \tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right)$$

$$= U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle\right) - U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right) \qquad \text{since } U_f \text{ is a linear operator}$$

$$= \tfrac{1}{\sqrt{2}}U_f\Big(|\mathbf{x}\rangle \otimes |0\rangle\Big) - \tfrac{1}{\sqrt{2}}U_f\Big(|\mathbf{x}\rangle \otimes |1\rangle\Big) \qquad \text{since } U_f \text{ is a linear operator}$$

$$= \tfrac{1}{\sqrt{2}}\Big(\boxed{|\mathbf{x}\rangle \otimes |f(\mathbf{x})\rangle}\Big) - \tfrac{1}{\sqrt{2}}\Big(\boxed{|\mathbf{x}\rangle \otimes |\overline{f(\mathbf{x})}\rangle}\Big)$$

When $|y\rangle$ is the superposition state $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, applying U_f gives:

$$
\begin{aligned}
& U_f\left(|\mathbf{x}\rangle \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\right) \\
&= U_f\left(|\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|0\rangle - |\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|1\rangle\right) \\
&= U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle - \tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right) \\
&= U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle\right) - U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right) && \text{since } U_f \text{ is a linear operator} \\
&= \tfrac{1}{\sqrt{2}}U_f\Big(|\mathbf{x}\rangle \otimes |0\rangle\Big) - \tfrac{1}{\sqrt{2}}U_f\Big(|\mathbf{x}\rangle \otimes |1\rangle\Big) && \text{since } U_f \text{ is a linear operator} \\
&= \tfrac{1}{\sqrt{2}}\Big(|\mathbf{x}\rangle \otimes |f(\mathbf{x})\rangle\Big) - \tfrac{1}{\sqrt{2}}\Big(|\mathbf{x}\rangle \otimes |\overline{f(\mathbf{x})}\rangle\Big) \\
&= \Big(|\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|f(\mathbf{x})\rangle\Big) - \Big(|\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|\overline{f(\mathbf{x})}\rangle\Big)
\end{aligned}
$$

When $|y\rangle$ is the superposition state $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, applying U_f gives:

$$
\begin{aligned}
& U_f\left(|\mathbf{x}\rangle \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\right) \\
&= U_f\left(|\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|0\rangle - |\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|1\rangle\right) \\
&= U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle - \tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right) \\
&= U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |0\rangle\right) - U_f\left(\tfrac{1}{\sqrt{2}}|\mathbf{x}\rangle \otimes |1\rangle\right) && \text{since } U_f \text{ is a linear operator} \\
&= \tfrac{1}{\sqrt{2}}U_f\Big(|\mathbf{x}\rangle \otimes |0\rangle\Big) - \tfrac{1}{\sqrt{2}}U_f\Big(|\mathbf{x}\rangle \otimes |1\rangle\Big) && \text{since } U_f \text{ is a linear operator} \\
&= \tfrac{1}{\sqrt{2}}\Big(|\mathbf{x}\rangle \otimes |f(\mathbf{x})\rangle\Big) - \tfrac{1}{\sqrt{2}}\Big(|\mathbf{x}\rangle \otimes |\overline{f(\mathbf{x})}\rangle\Big) \\
&= \left(|\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|f(\mathbf{x})\rangle\right) - \left(|\mathbf{x}\rangle \otimes \tfrac{1}{\sqrt{2}}|\overline{f(\mathbf{x})}\rangle\right) \\
&= |\mathbf{x}\rangle \otimes \left(\tfrac{1}{\sqrt{2}}|f(\mathbf{x})\rangle - \tfrac{1}{\sqrt{2}}|\overline{f(\mathbf{x})}\rangle\right)
\end{aligned}
$$

$$|\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|f(\mathbf{x})\rangle - \frac{1}{\sqrt{2}}|\overline{f(\mathbf{x})}\rangle\right)$$

$$= \begin{cases} |\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = |\mathbf{x}\rangle \otimes +\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) & \text{if } f(\mathbf{x}) = 0 \\ \\ |\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle\right) = |\mathbf{x}\rangle \otimes -\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) & \text{if } f(\mathbf{x}) = 1 \end{cases}$$

$$|\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|f(\mathbf{x})\rangle - \frac{1}{\sqrt{2}}|\overline{f(\mathbf{x})}\rangle\right)$$

$$= \begin{cases} |\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = |\mathbf{x}\rangle \otimes + \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) & \text{if } f(\mathbf{x}) = 0 \\ |\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle\right) = |\mathbf{x}\rangle \otimes - \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) & \text{if } f(\mathbf{x}) = 1 \end{cases}$$

$$= |\mathbf{x}\rangle \otimes (-1)^{f(\mathbf{x})}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

since $(-1)^{f(\mathbf{x})}$ is $+1$ when $f(\mathbf{x}) = 0$ and -1 when $f(\mathbf{x}) = 1$

$$= (-1)^{f(\mathbf{x})}|\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$(-1)^{f(\mathbf{x})}|\mathbf{x}\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right)$$