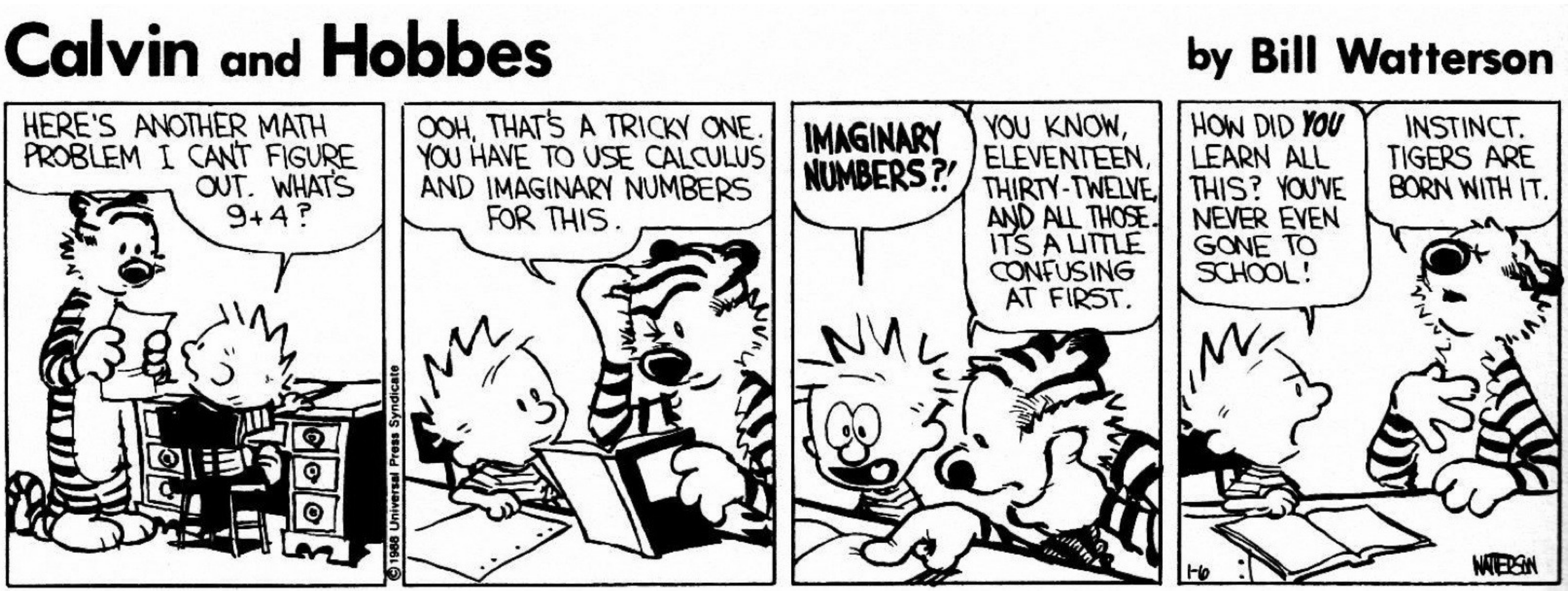
Background on Complex Numbers


Calvin and Hobbes
by Bill Watterson
HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE, AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.
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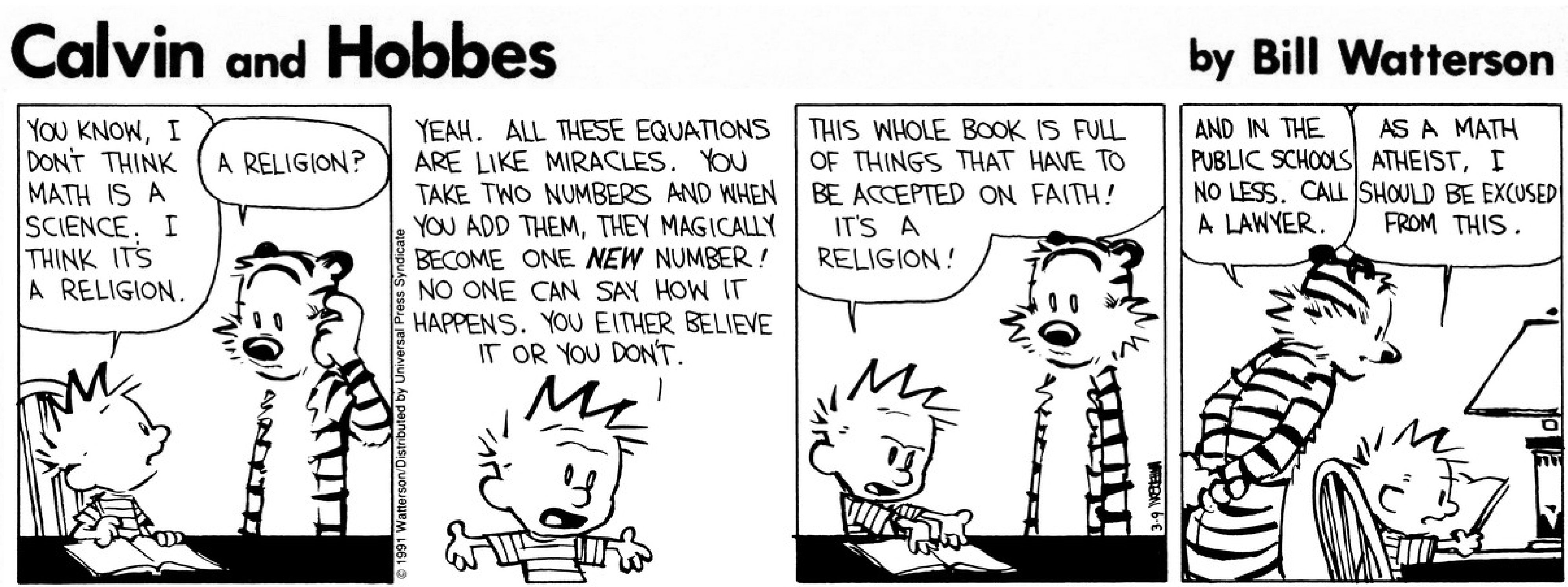
1-6
WATTERSON


Calvin and Hobbes
by Bill Watterson
YOU KNOW, I DON'T THINK MATH IS A SCIENCE. I THINK IT'S A RELIGION.
A RELIGION?
YEAH. ALL THESE EQUATIONS ARE LIKE MIRACLES. YOU TAKE TWO NUMBERS AND WHEN YOU ADD THEM, THEY MAGICALLY BECOME ONE NEW NUMBER! NO ONE CAN SAY HOW IT HAPPENS. YOU EITHER BELIEVE IT OR YOU DON'T.
THIS WHOLE BOOK IS FULL OF THINGS THAT HAVE TO BE ACCEPTED ON FAITH! IT'S A RELIGION!
AND IN THE PUBLIC SCHOOLS NO LESS. CALL A LAWYER.
AS A MATH ATHEIST, I SHOULD BE EXCUSED FROM THIS.
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The Concept of *Number*

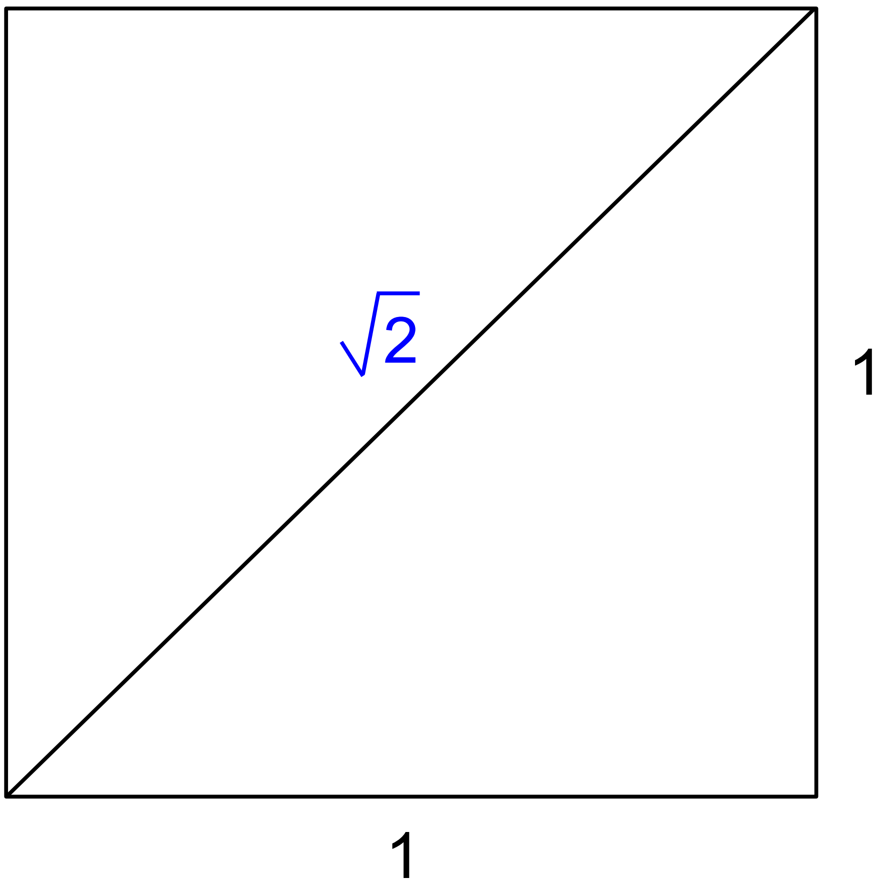
- The ancient Greeks understood the concept of *number* to mean
  - **Whole** numbers
    1, 2, 3, 4, ...
  - **Ratios** of whole numbers (fractions or rationals)
    1/2, 3/4, 1/4, 2/3, 3/4, 5/2, 6/5, 5/1, …
- These numbers covered all types of quantities in existence
- But then Pythagoras made a deeply shocking discovery:

Other types of numbers must exist!

- This knowledge was deemed too dangerous to divulge

The Concept of *Number*

- What is the length of the diagonal of this square?



By the Pythagorean Theorem:

$$\sqrt{1^2 + 1^2} = \sqrt{2}$$

The Concept of *Number*

- Pythagoras assumed that the number $\sqrt{2}$ in principle must be expressible as the **ratio of two whole numbers**

$$\sqrt{2} = \frac{a}{b} \qquad \text{with } \frac{a}{b} \text{ in } \textbf{lowest terms}$$

- Let's see where this assumption leads us ...
- Each step of our reasoning must be **absolutely convincing**
- **No faith is needed** (sorry, Calvin) — except for our assumption!

First, some basic number facts:

- **Fact 1:**

  Squaring a whole number always preserves **even/odd-ness**

  $3^2 = 9$ $5^2 = 25$ $15^2 = 225$

  $4^2 = 16$ $6^2 = 36$ $14^2 = 196$

- **Fact 2:**

  A fraction in lowest terms must contain at least one **odd number**

  1/3 and 3/4 cannot be reduced

  4/6 reduces to 2/3

  4/12 reduces to 2/6, which reduces to 1/3

(Proof that the square root of 2 is irrational)

Fast Forward to the 16th Century...

- **Irrational** numbers such as $\sqrt{2}$ and π are fully accepted
- **Negative** numbers such as -3 still make mathematicians squirm
- Some derisively call them “fictitious numbers”
- Cutting-edge research of the day: understanding and solving **cubic equations**

$$x^3 + 6x = 20 \qquad\qquad x^3 - 15x = 4$$

An Amusing Little Story

- **Scipione del Ferro**, mathematician (Bologna)
- **Antonio Fior**, student of del Ferro
- **Niccolo Tartaglia**, mathematician (Brescia)
- **Gerolamo Cardano**, physician, mathematician, philosopher, gambler, and all around Renaissance man (Milan)

Tartaglia

Cardano

Around 1500, **Scipione del Ferro** discovered how to solve the "depressed" cubic equation:

$$x^3 + px = q$$

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

But he kept this knowledge a **closely-guarded secret** all his life

Example

$$x^3 + 6x = 20$$

$$p = 6 \qquad q = 20$$

$$x = \sqrt[3]{\frac{20}{2} + \sqrt{\frac{20^2}{4} + \frac{6^3}{27}}} - \sqrt[3]{-\frac{20}{2} + \sqrt{\frac{20^2}{4} + \frac{6^3}{27}}}$$

$$= \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

$$= \ 2$$

On his deathbed, del Ferro confided the secret to his student **Antonio Fior**, who was a very mediocre mathematician

In 1535, Fior challenged **Niccolo Tartaglia** to a public contest

...and **lost badly**, because Tartaglia re-discovered del Ferro's solution for himself just before the contest

Gerolamo Cardano also re-discovered del Ferro's solution, and published it in his book *Ars Magna* in 1545

His book had 13 chapters, one for each "type" of cubic equation

$x^3 + px = q$ $\qquad$ $x^3 + 6x = 20$

$x^3 = px + q$ $\qquad$ $x^3 = 15x + 4$

$x^3 + px^2 = q$ $\qquad$ $x^3 + 2x^2 = 16$

etc.

This was a much greater achievement than del Ferro's single formula (which is now called "Cardano's formula")

But There Was Still a Mystery

$$x^3 - 15x = 4$$

It's easy to see that the solution is $x = 4$

But There Was Still a Mystery

$$x^3 - 15x = 4$$

Solving this using Cardano's formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}$$

But how could this possibly be equivalent to 4 ???

In fact, all three roots of the equation are clearly **real**:

$x = 4$ $\quad x = -2 + \sqrt{3}$ $\quad x = -2 - \sqrt{3}$

Cardano called such cubic equations “irreducible”

But There Was Still a Mystery

$$x^3 - 15x = 4$$

In subsequent work, Rafael Bombelli (1526-72) was able to **prove** that

$$\sqrt[3]{2+\sqrt{-121}} - \sqrt[3]{-2+\sqrt{-121}}$$

really is the ordinary number 4.

This was one of the first clues that eventually forced mathematicians to (grudgingly) accept that square roots of negative numbers might really be legitimate.

But There Was Still a Mystery

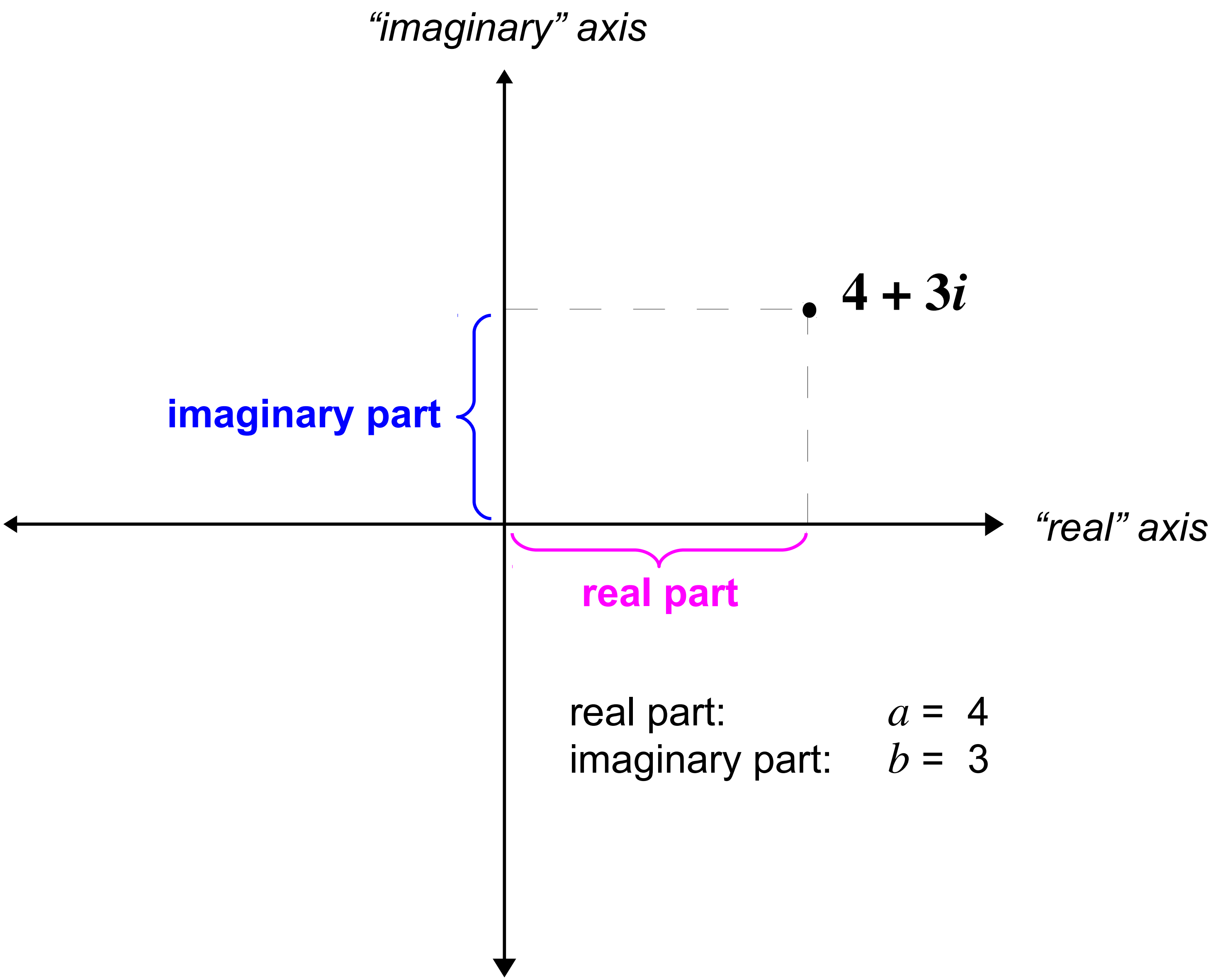
[Square roots of negative numbers] are not nothing, nor less than nothing, which makes them imaginary, indeed impossible

—Leonhard Euler

In mathematics, you don't understand things. You just get used to them.

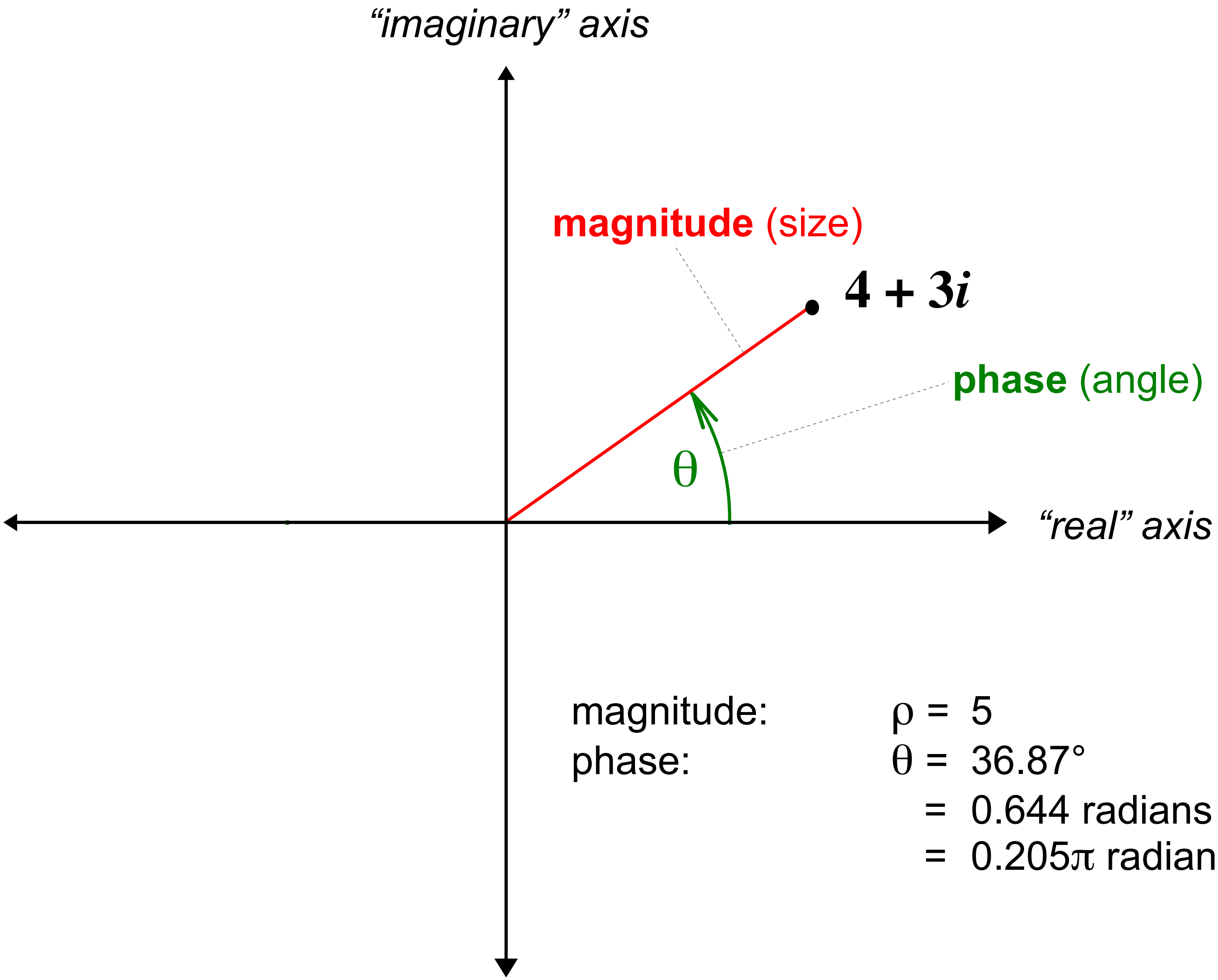
—John von Neumann

Complex Numbers



real part: a = 4
imaginary part: b = 3

Complex Numbers



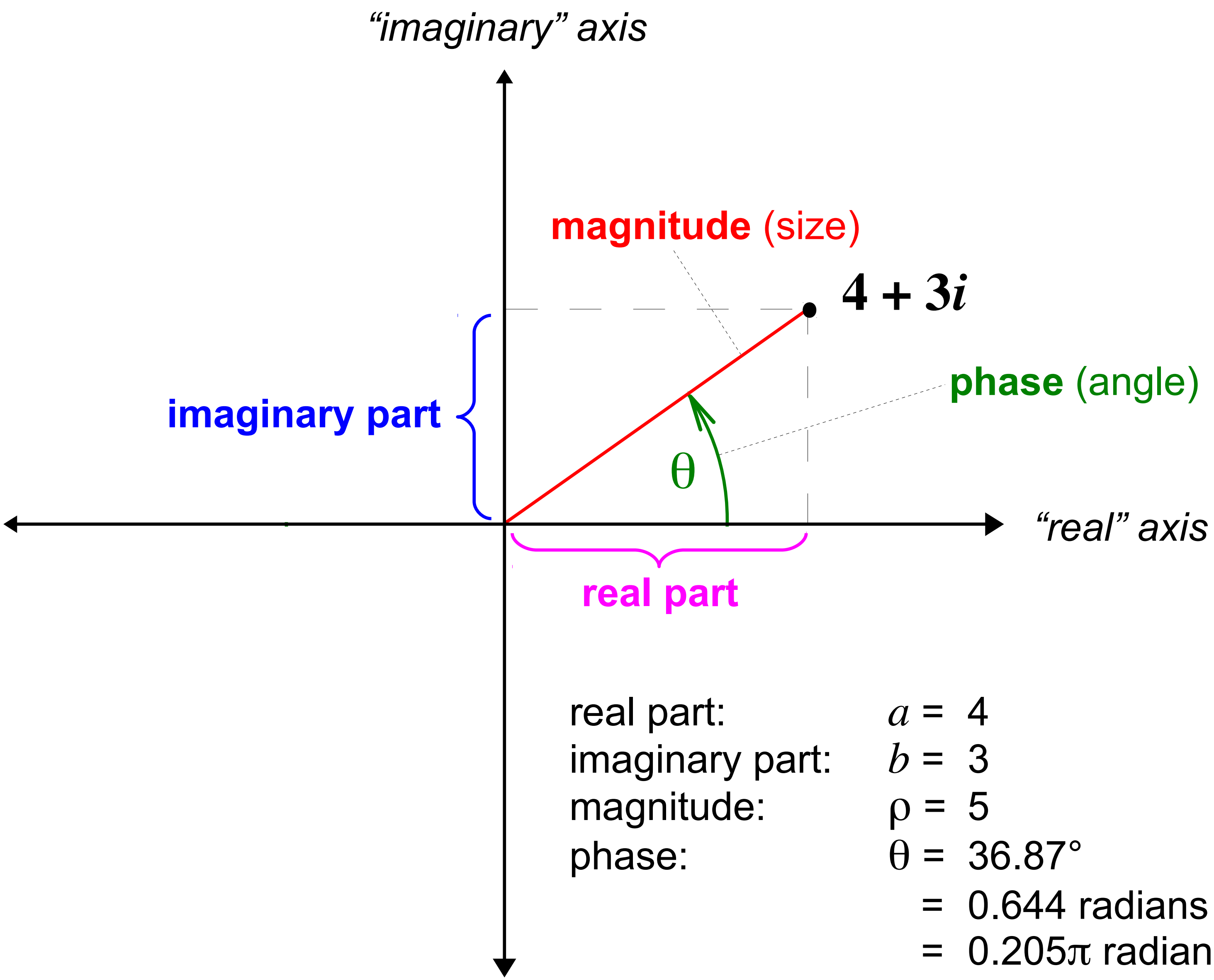
magnitude: $\rho = 5$

phase: $\theta = 36.87°$

$= 0.644$ radians

$= 0.205\pi$ radians

Complex Numbers



real part: $a = 4$

imaginary part: $b = 3$

magnitude: $\rho = 5$

phase: $\theta = 36.87°$

$= 0.644$ radians

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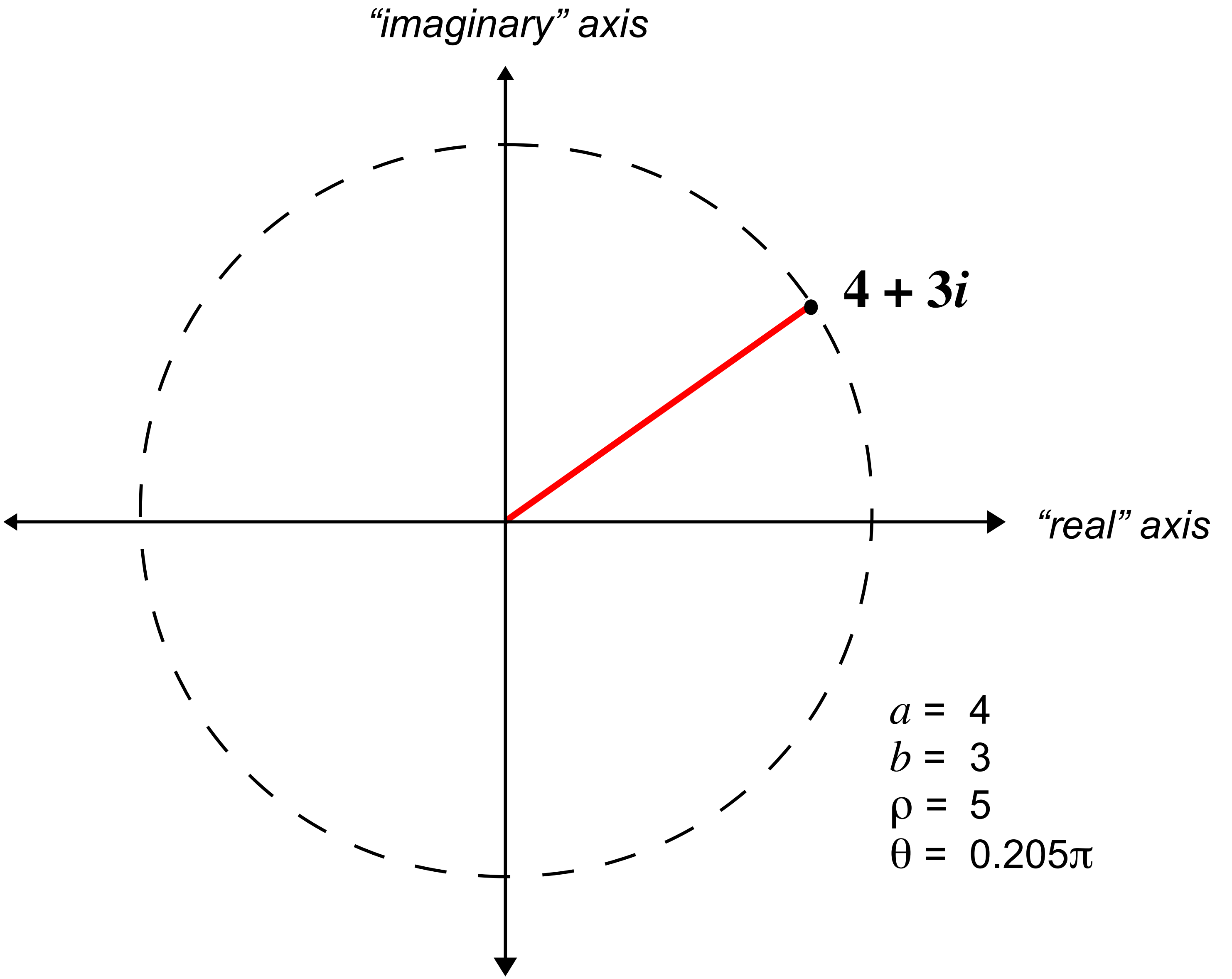
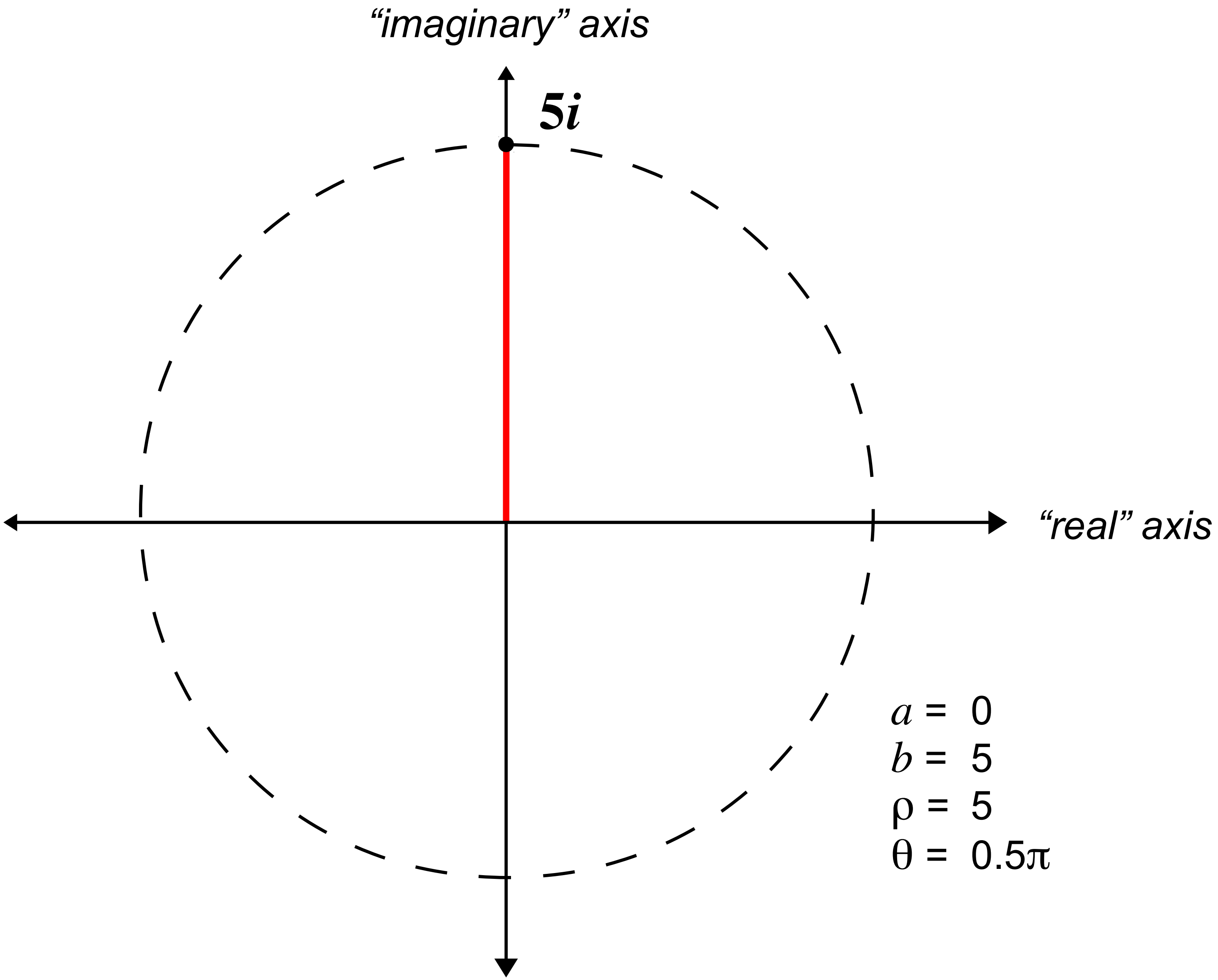
Complex Numbers

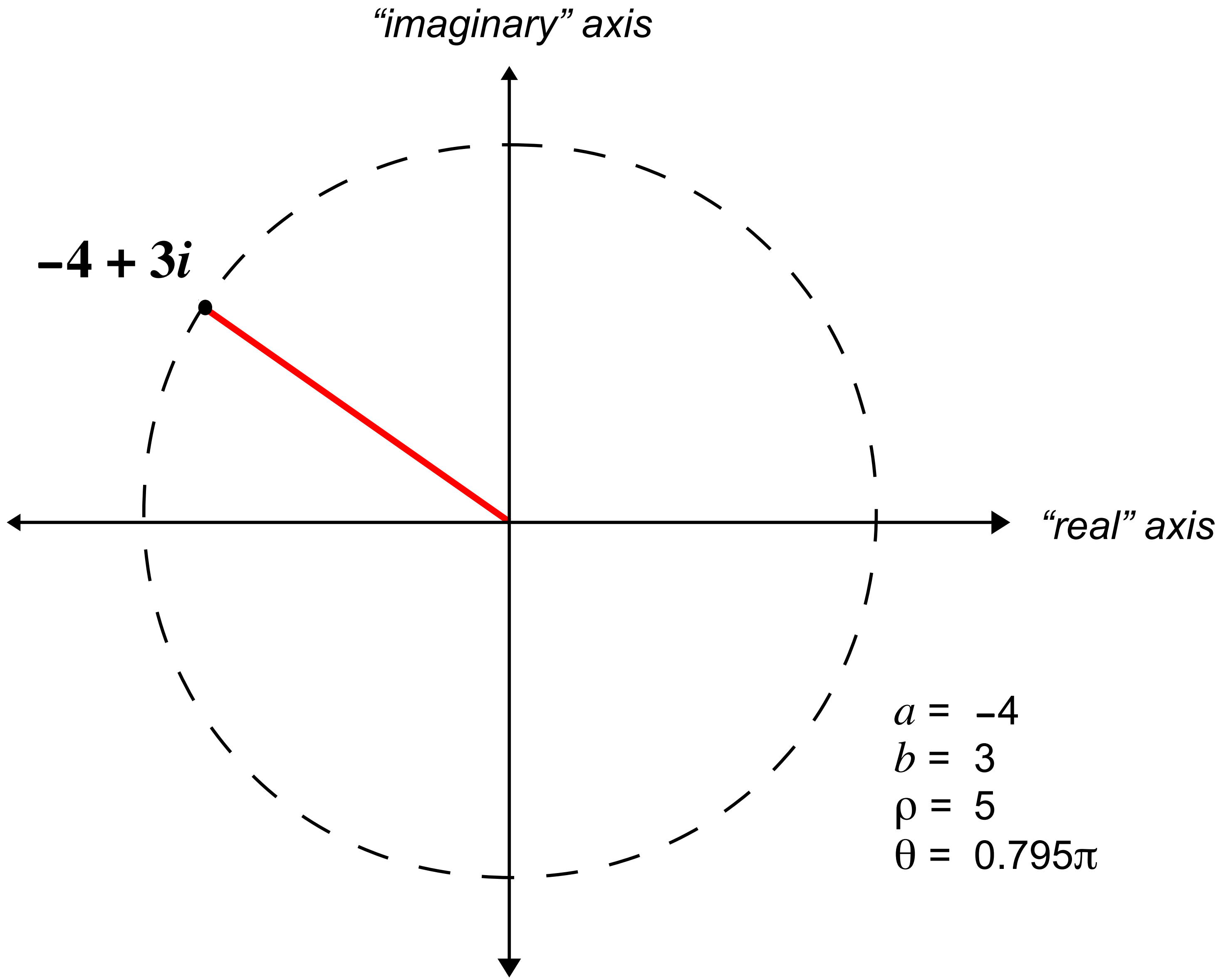


Complex Numbers



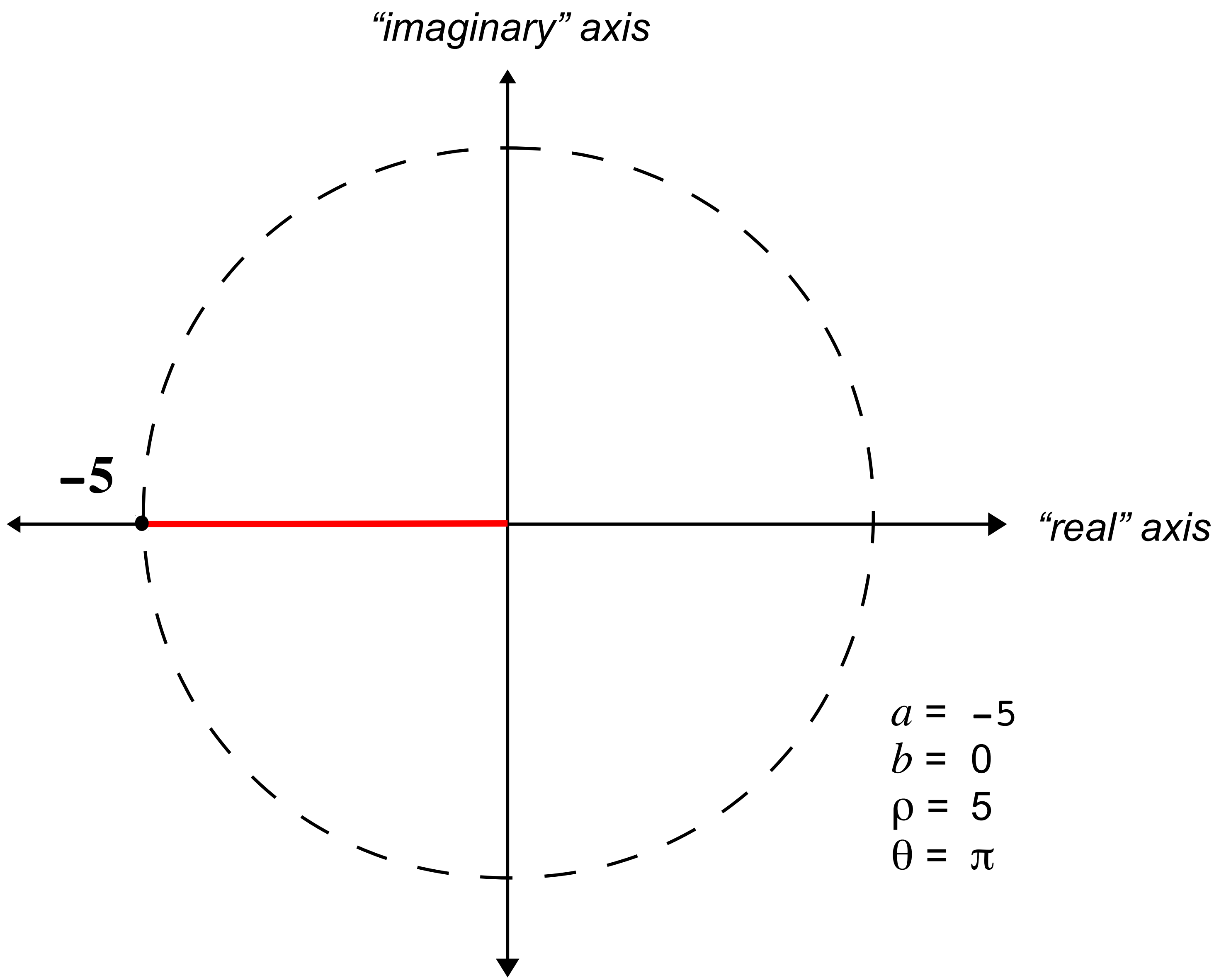
a = 0
b = 5
ρ = 5
θ = 0.5π

Complex Numbers



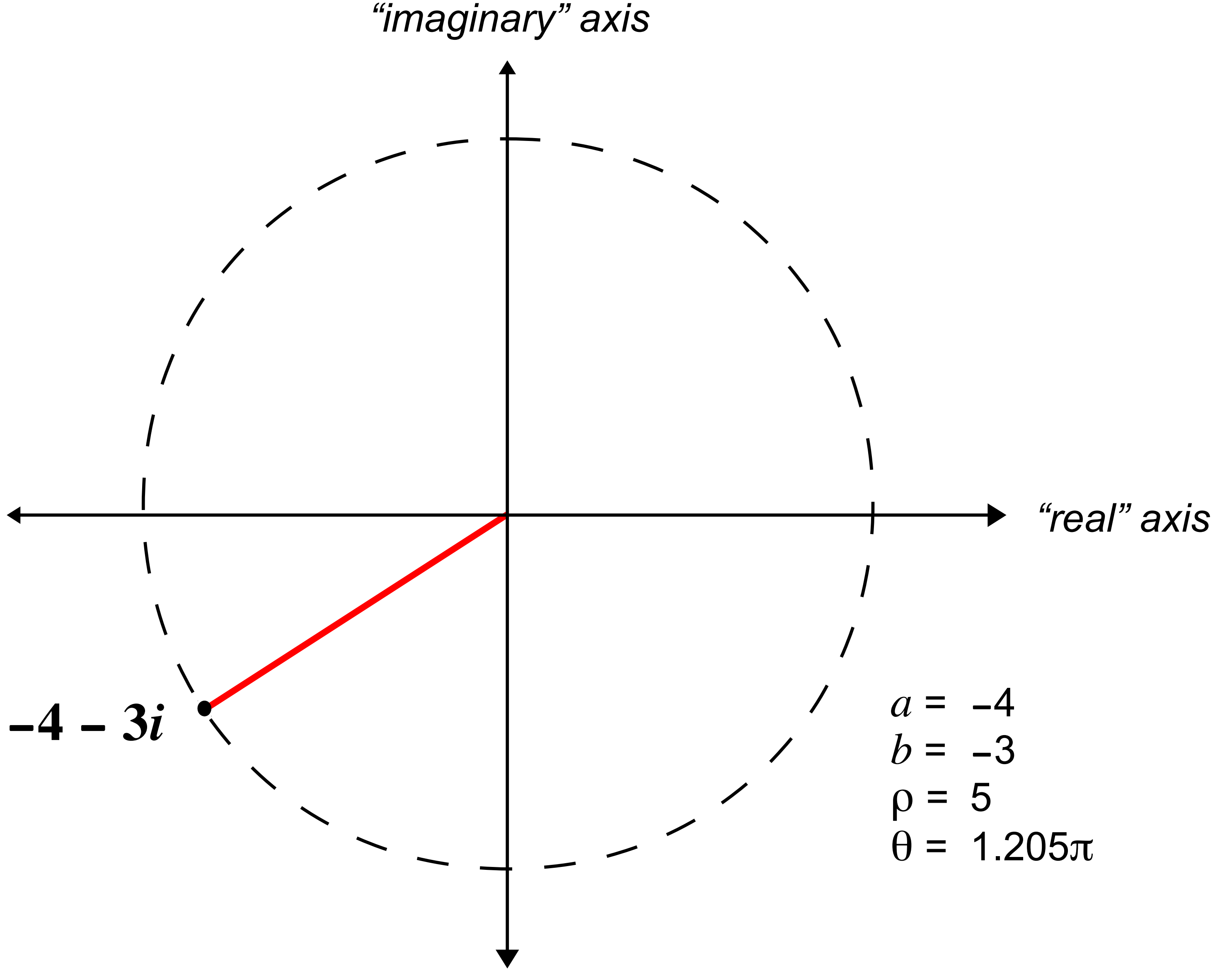
$a = -4$
$b = 3$
$\rho = 5$
$\theta = 0.795\pi$

Complex Numbers

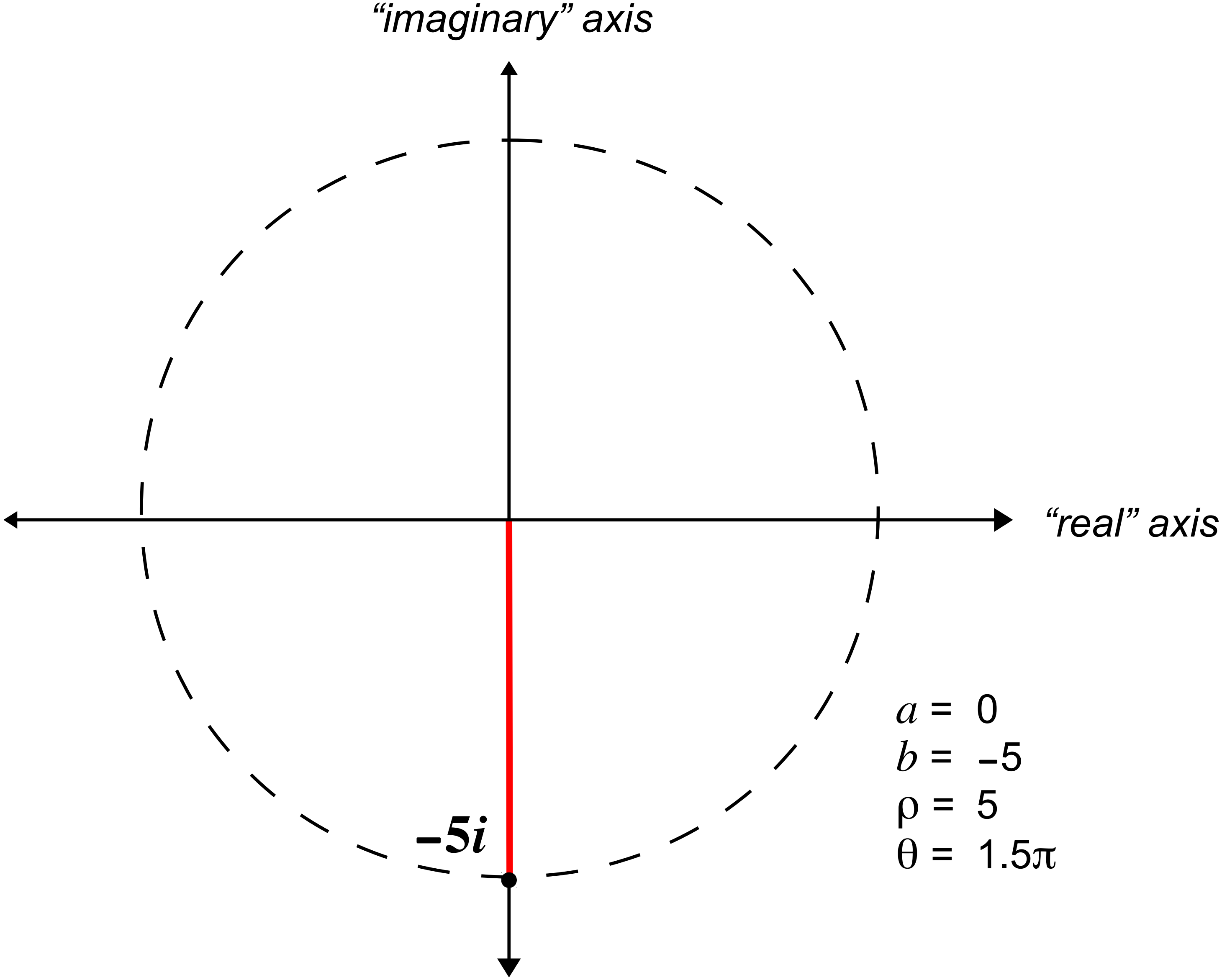


$a = -5$
$b = 0$
$\rho = 5$
$\theta = \pi$

Complex Numbers

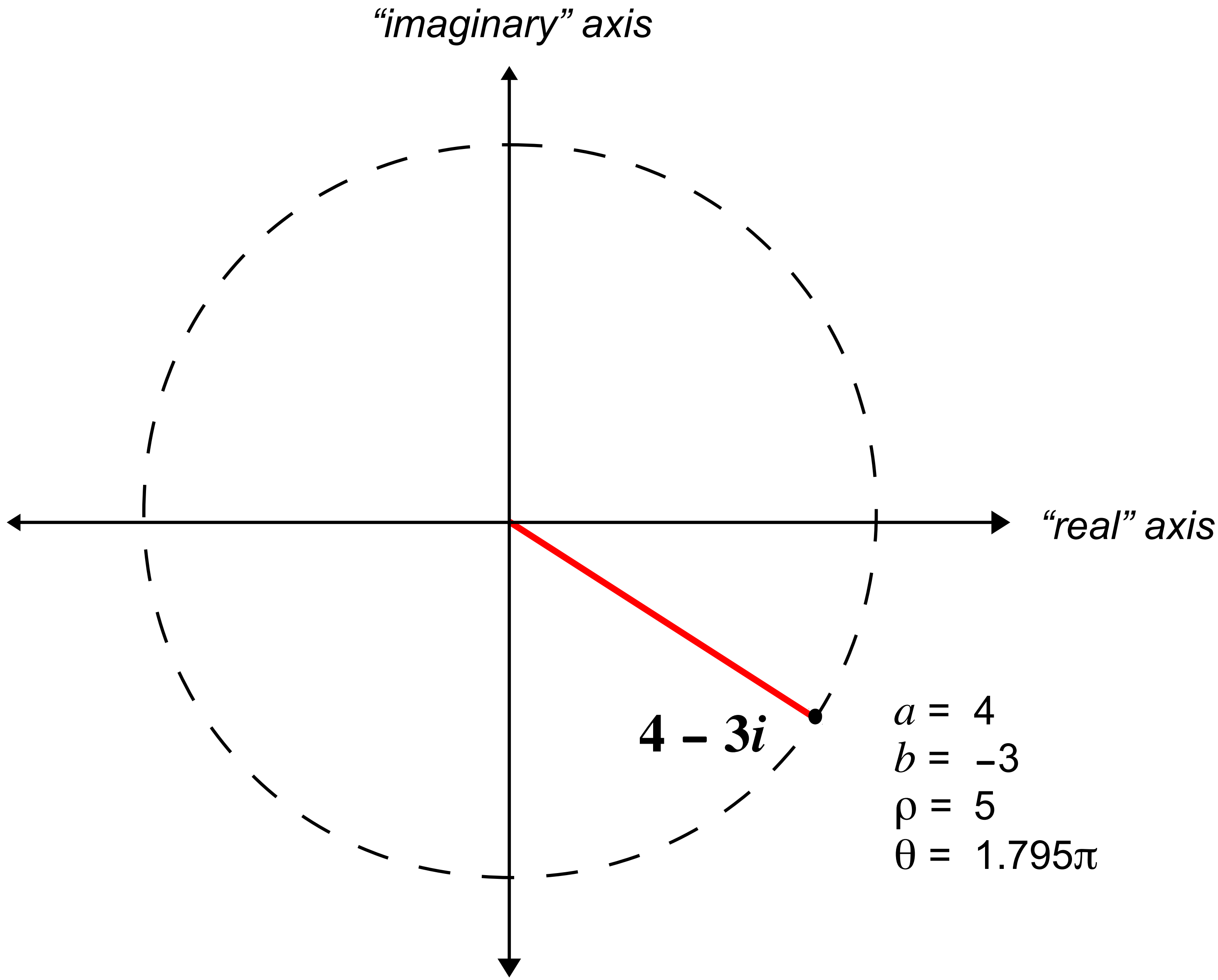


Complex Numbers



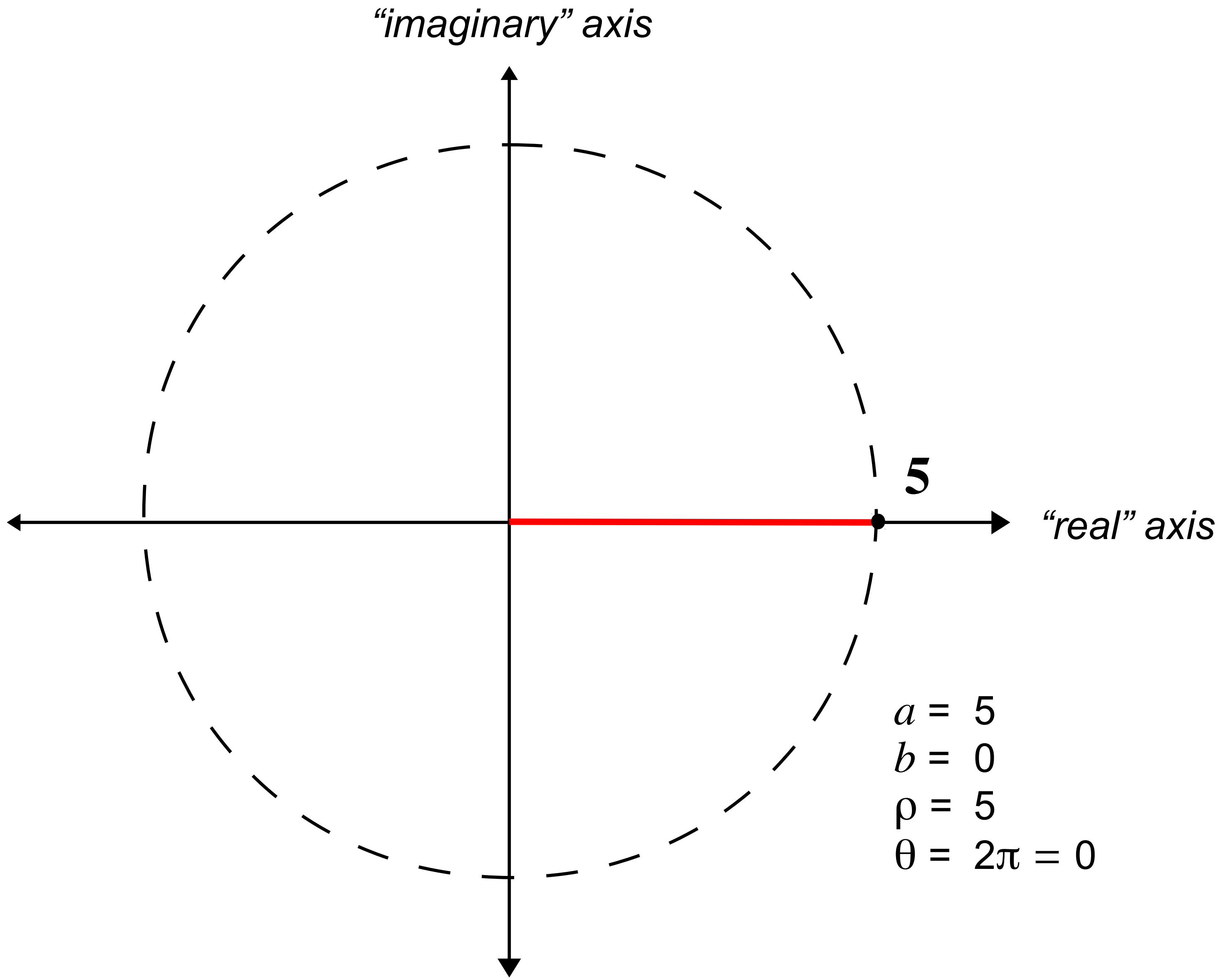
$a =$ 0
$b =$ −5
$\rho =$ 5
$\theta =$ 1.5π

Complex Numbers



a = 4
b = −3
ρ = 5
θ = 1.795π

Complex Numbers



a = 5
b = 0
ρ = 5
θ = $2\pi = 0$