Combining qubit operations

Applying the single-qubit operator H to the qubit $|0\rangle$ looks like this, in terms of matrices:

$$H|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

Likewise, applying the single-qubit operator I to $|1\rangle$ looks like this:

$$I|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

The tensor product $\otimes$ allows us to combine two single-qubit operators such as H and I into one 2-qubit operator, which can be applied to two qubits at once. For example, in terms of matrices, the combined operator $H \otimes I$ looks like this:

$$H \otimes I = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

or, equivalently:

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

We can apply the $H \otimes I$ operator to the qubits $|0\rangle$ and $|1\rangle$ like this:

$$(H \otimes I)(|0\rangle \otimes |1\rangle) = (H \otimes I)|01\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) + \frac{1}{\sqrt{2}}(|1\rangle \otimes |1\rangle)$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |1\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |1\rangle$$

which is exactly what we got by performing the operations individually.

Tensor product rules

- $(A \otimes B \otimes C \ldots)(X \otimes Y \otimes Z \ldots) = AX \otimes BY \otimes CZ \ldots$
- $(A+B) \otimes X = A \otimes X + B \otimes X$ and $X \otimes (A+B) = X \otimes A + X \otimes B$
- $(A+B) \otimes (X+Y) = A \otimes X + A \otimes Y + B \otimes X + B \otimes Y$
- $\alpha A \otimes \beta B = A \otimes \alpha\beta B = \alpha\beta A \otimes B = \alpha\beta(A \otimes B)$

H applied to $|0\rangle$

$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$H \otimes H$ applied to $|00\rangle$

$$(H \otimes H)(|0\rangle \otimes |0\rangle)$$

$$= H|0\rangle \otimes H|0\rangle$$

$$= \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}}|0\rangle \otimes \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|0\rangle \otimes \tfrac{1}{\sqrt{2}}|1\rangle + \tfrac{1}{\sqrt{2}}|1\rangle \otimes \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle \otimes \tfrac{1}{\sqrt{2}}|1\rangle$$

$$= \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle\Big) + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|0\rangle \otimes |1\rangle\Big) + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|1\rangle \otimes |0\rangle\Big) + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|1\rangle \otimes |1\rangle\Big)$$

$$= \tfrac{1}{\sqrt{4}}|00\rangle + \tfrac{1}{\sqrt{4}}|01\rangle + \tfrac{1}{\sqrt{4}}|10\rangle + \tfrac{1}{\sqrt{4}}|11\rangle$$

$H \otimes H \otimes H$ applied to $|000\rangle$

$$(H \otimes H \otimes H)(|0\rangle \otimes |0\rangle \otimes |0\rangle)$$

$$= H|0\rangle \otimes H|0\rangle \otimes H|0\rangle$$

$$= \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left[\tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle\Big) + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|0\rangle \otimes |1\rangle\Big) + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|1\rangle \otimes |0\rangle\Big) + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|1\rangle \otimes |1\rangle\Big)\right]$$

$$= \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle \otimes |0\rangle\Big) + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle \otimes |1\rangle\Big) + \ldots + \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\Big(|1\rangle \otimes |1\rangle \otimes |1\rangle\Big)$$

$$= \tfrac{1}{\sqrt{8}}|000\rangle + \tfrac{1}{\sqrt{8}}|001\rangle + \tfrac{1}{\sqrt{8}}|010\rangle + \tfrac{1}{\sqrt{8}}|011\rangle + \tfrac{1}{\sqrt{8}}|100\rangle + \tfrac{1}{\sqrt{8}}|101\rangle + \tfrac{1}{\sqrt{8}}|110\rangle + \tfrac{1}{\sqrt{8}}|111\rangle$$

$H \otimes H \otimes \ldots \otimes H$ applied to $|00\ldots0\rangle$

$$H^{\otimes n}|0^n\rangle \;=\; \sum_{\mathbf{x} \in \{0,1\}^n} \frac{1}{\sqrt{2^n}}\,|\mathbf{x}\rangle \;=\; \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x} \in \{0,1\}^n} |\mathbf{x}\rangle$$