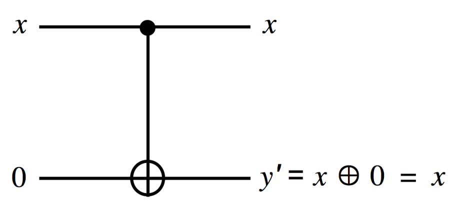
Entangling qubits with CNOT

The CNOT gate with the y input set to 0 works fine for copying the classical bits 0 and 1:



$$CNOT\Big(|0\rangle \otimes |0\rangle\Big) = |0\rangle \otimes |0\rangle \qquad CNOT\Big(|1\rangle \otimes |0\rangle\Big) = |1\rangle \otimes |1\rangle$$

$$CNOT|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \qquad CNOT|10\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

But what happens if the top input qubit is in an arbitrary superposition state $\alpha|0\rangle + \beta|1\rangle$ instead? If the CNOT gate correctly copies this qubit, we would *expect* that the state of each output qubit would be equal to $\alpha|0\rangle + \beta|1\rangle$. That is:

$$\begin{aligned} CNOT\Big(\big(\alpha|0\rangle + \beta|1\rangle\big) \otimes |0\rangle\Big) \text{ should give } & \big(\alpha|0\rangle + \beta|1\rangle\big) \otimes \big(\alpha|0\rangle + \beta|1\rangle\big) \\ &= \alpha|0\rangle \otimes \alpha|0\rangle + \alpha|0\rangle \otimes \beta|1\rangle + \beta|1\rangle \otimes \alpha|0\rangle + \beta|1\rangle \otimes \beta|1\rangle \\ &= \alpha^2\big(|0\rangle \otimes |0\rangle\big) + \alpha\beta\big(|0\rangle \otimes |1\rangle\big) + \beta\alpha\big(|1\rangle \otimes |0\rangle\big) + \beta^2\big(|1\rangle \otimes |1\rangle\big) \\ &= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle \end{aligned}$$

But that is not what actually happens. Instead of being in the above separable state, with both output qubits in identical superpositions of $|0\rangle$ and $|1\rangle$, the two qubits end up in an *entangled* state:

$$CNOT\Big(\big(\alpha|0\rangle + \beta|1\rangle\big) \otimes |0\rangle\Big) = CNOT\Big(\alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |0\rangle\Big)$$

$$= CNOT\Big(\alpha|00\rangle + \beta|10\rangle\Big) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{bmatrix} = \alpha|00\rangle + \beta|11\rangle$$

We cannot factor this 2-qubit state into the tensor product of two 1-qubit states. Thus the output qubits are *not* two independent copies of the input qubit. Since they are entangled, in a certain sense they could be regarded as being literally *the same qubit*. As soon as we measure one of them, yielding either $|0\rangle$ with probability $|\alpha|^2$ or $|1\rangle$ with probability $|\beta|^2$, the other one immediately acquires the same state with 100% certainty. At that moment, the entanglement is broken, and both qubits become independent, classical bits from that point on.