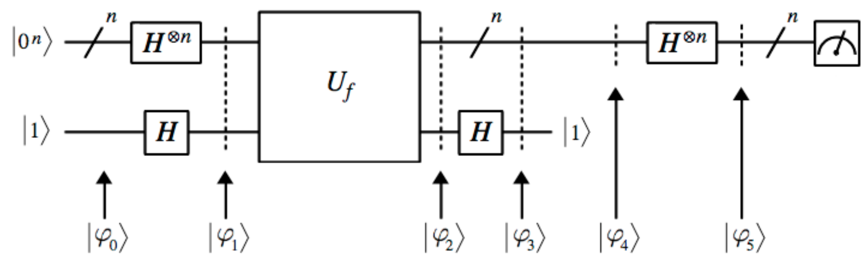
Deutsch-Jozsa algorithm (Marshall's variation)



Now let's generalize x to be an arbitrary n-bit binary string, which we will write as $\mathbf{x}$. Whereas $|x\rangle$ represented a single qubit, $|\mathbf{x}\rangle$ will represent an n-qubit quantum register. Instead of creating an equal superposition of $|0\rangle$ and $|1\rangle$ as before, we will create an equal superposition of all n-bit binary strings $|000\ldots0\rangle$ to $|111\ldots1\rangle$ using n Hadamard gates applied to $|000\ldots0\rangle$:

$$H^{\otimes n}|0^n\rangle = (H \otimes H \otimes \ \ldots \ \otimes H)|00\ldots0\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle$$

We apply U_f to the n-qubit superposition $H^{\otimes n}|0^n\rangle$, which represents the binary codes of all of the integers from 0 to $2^n - 1$ simultaneously, and the 1-qubit superposition $H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$:

$$|\varphi_0\rangle = |0^n\rangle \otimes |1\rangle$$

$$\begin{aligned} |\varphi_1\rangle &= H^{\otimes n}|0^n\rangle \otimes H|1\rangle \\ &= \left(\frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) \end{aligned}$$

Applying U_f to $|\varphi_1\rangle$ will produce the output:

$$|\varphi_2\rangle = \left(\frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})}|\mathbf{x}\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

We can turn the second qubit back into $|1\rangle$ by applying a single H gate to it:

$$|\varphi_3\rangle = \left(\frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})}|\mathbf{x}\rangle\right) \otimes |1\rangle$$

Since we're only interested in the top n qubits, we will just ignore the bottom qubit:

$$|\varphi_4\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})}|\mathbf{x}\rangle$$

We then apply n Hadamard gates to the top n qubits and measure the result. If we obtain 0^n, then f is constant. If we obtain anything else, then f is balanced (assuming that f was either constant or balanced to begin with). If f is neither constant nor balanced, the measurement will not yield reliable information.

Some examples

To make things more concrete, consider the case when $n = 2$, that is, when $|\mathbf{x}\rangle$ consists of 2 qubits. The top two output qubits after applying U_f are then:

$$|\varphi_4\rangle = \tfrac{1}{2}\Big((-1)^{f(00)}|00\rangle + (-1)^{f(01)}|01\rangle + (-1)^{f(10)}|10\rangle + (-1)^{f(11)}|11\rangle\Big)$$

Notice that the + or − signs for $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ are determined by the behavior of f on each of the $\mathbf{x}$ values 00, 01, 10, and 11, according to the term $(-1)^{f(\mathbf{x})}$.

- If f is the "constant 0" function, $f(\mathbf{x}) = 0$ for all values of $\mathbf{x}$, which makes all of the $(-1)^{f(\mathbf{x})}$ coefficients equal to +1:

  $$|\varphi_4\rangle = \tfrac{1}{2}\Big(+|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big)$$

  This state is equivalent to $(H \otimes H)|00\rangle$. Applying the 2-qubit operator $H \otimes H$ to this state in effect removes the $(H \otimes H)$, giving $|00\rangle$ as the final state $|\varphi_5\rangle$. When we measure the final state, we will get 00 with 100% certainty, indicating that f is constant.

- If f is the "constant 1" function, $f(\mathbf{x}) = 1$ for all values of $\mathbf{x}$, which makes all of the $(-1)^{f(\mathbf{x})}$ coefficients equal to −1:

  $$|\varphi_4\rangle = \tfrac{1}{2}\Big(-|00\rangle - |01\rangle - |10\rangle - |11\rangle\Big)$$

  This state is equivalent to $(H \otimes H)(-|00\rangle)$. Applying $H \otimes H$ to this state gives $-|00\rangle$ as the final state $|\varphi_5\rangle$. When we measure the final state, we will get 00 with 100% certainty, indicating that f is constant.

- If f is the balanced function $00 \to 1, 01 \to 1, 10 \to 0, 11 \to 0$, we get:

  $$|\varphi_4\rangle = \tfrac{1}{2}\Big(-|00\rangle - |01\rangle + |10\rangle + |11\rangle\Big)$$

  This state is equivalent to $(H \otimes H)(-|10\rangle)$. Applying $H \otimes H$ to this state gives $-|10\rangle$ as the final state $|\varphi_5\rangle$. When we measure the final state, we will get 10 with 100% certainty, indicating that f is balanced (since the outcome *wasn't* 00).

- If f is the balanced function $00 \to 0, 01 \to 1, 10 \to 0, 11 \to 1$, we get:

  $$|\varphi_4\rangle = \tfrac{1}{2}\Big(+|00\rangle - |01\rangle + |10\rangle - |11\rangle\Big)$$

  This state is equivalent to $(H \otimes H)|01\rangle$. Applying $H \otimes H$ to this state gives $|01\rangle$ as the final state $|\varphi_5\rangle$. When we measure the final state, we will get 01 with 100% certainty, indicating that f is balanced (since the outcome *wasn't* 00).

- If f is the *unbalanced* function $00 \to 0, 01 \to 0, 10 \to 0, 11 \to 1$, we get:

  $$|\varphi_4\rangle = \tfrac{1}{2}\Big(+|00\rangle + |01\rangle + |10\rangle - |11\rangle\Big)$$

  Applying $H \otimes H$ to this state just gives us back the same state $\big(\tfrac{1}{2}|00\rangle + \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|10\rangle - \tfrac{1}{2}|11\rangle\big)$ for the final state $|\varphi_5\rangle$. When we measure the final state, we will get one of 00, 01, 10, 11 with 25% probability, but the outcome won't convey any useful information about f.