Quantum state vectors

Suppose our state vector is $|\psi\rangle = [2-i, 2i, 1-i, 1, -2i, 2]^T$. How many basis states are there? Calculate the probabilities of finding the system in each basis state.

This is a 6-dimensional system, so there are 6 basis states, which we can label $|b_0\rangle$ through $|b_5\rangle$. The squared magnitudes of the amplitudes of $|\psi\rangle$ are:

$|2-i|^2 = 5$, $|2i|^2 = 4$, $|1-i|^2 = 2$, $|1|^2 = 1$, $|-2i|^2 = 4$, and $|2|^2 = 4$

and the sum of the squared magnitudes $\sum |c_i|^2 = 5+4+2+1+4+4 = 20$

An alternative way of computing the sum of the squared magnitudes is to use the inner product:

$$\langle \psi \,|\, \psi \rangle = \begin{bmatrix} (2-i)^* & (2i)^* & (1-i)^* & 1^* & (-2i)^* & 2^* \end{bmatrix} \begin{bmatrix} 2-i \\ 2i \\ 1-i \\ 1 \\ -2i \\ 2 \end{bmatrix}$$

$$= (2+i)(2-i) + (-2i)(2i) + (1+i)(1-i) + (1)(1) + (2i)(-2i) + (2)(2)$$

$$= 5+4+2+1+4+4 = 20$$

which is the same thing as the norm squared, since $\|\psi\| = \sqrt{\langle \psi \,|\, \psi \rangle}$

So the probabilities of finding the system, respectively, in each basis state when the system is observed are:

$P(b_0) = \frac{5}{20}$, $P(b_1) = \frac{4}{20}$, $P(b_2) = \frac{2}{20}$, $P(b_3) = \frac{1}{20}$, $P(b_4) = \frac{4}{20}$, and $P(b_5) = \frac{4}{20}$

which, reduced to lowest terms, gives:

$P(b_0) = \frac{1}{4}$, $P(b_1) = \frac{1}{5}$, $P(b_2) = \frac{1}{10}$, $P(b_3) = \frac{1}{20}$, $P(b_4) = \frac{1}{5}$, and $P(b_5) = \frac{1}{5}$

In general, given any state vector $|\psi\rangle$ expressed as a linear combination of the basis states $|b_i\rangle$ with amplitudes c_i:

$$|\psi\rangle = c_0 \,|b_0\rangle + c_1 \,|b_1\rangle + \ldots + c_{n-1} \,|b_{n-1}\rangle$$

we can calculate the probability that the system, when observed, will be found in basis state b_i:

$$P(b_i) = \frac{|c_i|^2}{|c_0|^2 + |c_1|^2 + \ldots + |c_{n-1}|^2} = \frac{|c_i|^2}{\sum |c_i|^2} = \frac{|c_i|^2}{\langle \psi \,|\, \psi \rangle}$$

Why size does not matter

Suppose we start with a "bigger" version of our state vector:

$7|\psi\rangle = [14-7i, 14i, 7-7i, 7, -14i, 14]^T$

What are the probabilities of observing each basis state now?

The squared magnitudes are now: $|14-7i|^2 = 245$, $|14i|^2 = 196$, $|7-7i|^2 = 98$, $|7|^2 = 49$, $|-14i|^2 = 196$, and $|14|^2 = 196$, so the sum of the squared magnitudes is $245+196+98+49+196+196 = 980$, which is 7^2 times the previous sum of 20. Thus, the probabilities are:

$P(b_0) = \frac{245}{980}$, $P(b_1) = \frac{196}{980}$, $P(b_2) = \frac{98}{980}$, $P(b_3) = \frac{49}{980}$, $P(b_4) = \frac{196}{980}$, and $P(b_5) = \frac{196}{980}$

which, when reduced to lowest terms, gives:

$P(b_0) = \frac{1}{4}$, $P(b_1) = \frac{1}{5}$, $P(b_2) = \frac{1}{10}$, $P(b_3) = \frac{1}{20}$, $P(b_4) = \frac{1}{5}$, and $P(b_5) = \frac{1}{5}$

which are exactly the same probabilities as before. So scaling the state vector has no effect on the observed probabilities. In general, *all* vectors $c|\psi\rangle$ for any $c \in \mathbb{C}$ represent *exactly the same physical quantum state*, as far as we the observers are concerned. So we might as well choose $|\psi\rangle$ to be normalized, so that $\|\psi\| = \sqrt{\langle\psi\,|\,\psi\rangle} = 1$, which makes calculating the probabilities much easier:

$P(b_i) = |c_i|^2$

An equivalent way of calculating the probabilities from a normalized state vector is to compute $|\langle b_i\,|\,\psi\rangle|^2$. Since $\langle b_i|$ is just $[0, 0, \ldots, 1, \ldots, 0]$, with the single 1 in the i^{th} position, calculating $\langle b_i\,|\,\psi\rangle$ just "grabs" the i^{th} amplitude from $|\psi\rangle$. We then square its magnitude to obtain the probability of the system transitioning from $|\psi\rangle$ to $|b_i\rangle$ when observed.

Normalizing a state vector

We can normalize $|\psi\rangle$ by simply dividing it by its length (norm), which we calculated before as $\|\psi\| = \sqrt{\langle\psi\,|\,\psi\rangle} = \sqrt{20}$.

So the normalized $|\psi'\rangle = \dfrac{|\psi\rangle}{\|\psi\|} = \dfrac{|\psi\rangle}{\sqrt{\langle\psi\,|\,\psi\rangle}} = \dfrac{[2-i, 2i, 1-i, 1, -2i, 2]^T}{\sqrt{20}}$

$|\psi'\rangle = [\frac{2-i}{\sqrt{20}}, \frac{2i}{\sqrt{20}}, \frac{1-i}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{-2i}{\sqrt{20}}, \frac{2}{\sqrt{20}}]^T$

From this it is easy to calculate the probabilities of observing each basis state. For example:

$$P(b_0) = \left|\frac{2-i}{\sqrt{20}}\right|^2 = \frac{|2-i|^2}{20} = \frac{5}{20} = \frac{1}{4}$$

Expressing a state vector $|\psi\rangle$ in the eigenbasis of an observable Ω

Suppose we have an observable Ω with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ and associated orthonormal eigenbasis vectors $|e_0\rangle, |e_1\rangle, \ldots, |e_{n-1}\rangle$. We can express any normalized state vector $|\psi\rangle$ "in this eigenbasis" by writing it explicitly as a linear combination of the basis vectors:

$|\psi\rangle = c_0|e_0\rangle + c_1|e_1\rangle + \ldots + c_{n-1}|e_{n-1}\rangle$

where the sum of the squared magnitudes $|c_0|^2 + |c_1|^2 + \ldots + |c_{n-1}|^2 = 1$.

However, if the state vector $|\psi\rangle$ is written in the conventional way as $[z_0, z_1, \ldots, z_{n-1}]^T$, the z_i's are generally *not* the amplitudes of Ω's eigenbasis vectors. In fact, the z_i's are the amplitudes of the standard canonical basis vectors $[1, 0, 0, \ldots, 0]^T$, $[0, 1, 0, \ldots, 0]^T$, etc. To express $|\psi\rangle$ in the eigenbasis of Ω, we first need to compute the amplitudes of the eigenbasis vectors using the inner product:

$c_0 = \langle e_0 \,|\, \psi\rangle$
$c_1 = \langle e_1 \,|\, \psi\rangle$
$\ldots$
$c_{n-1} = \langle e_{n-1} \,|\, \psi\rangle$

We can then rewrite the state vector in the eigenbasis as $|\psi\rangle = c_0|e_0\rangle + c_1|e_1\rangle + \ldots + c_{n-1}|e_{n-1}\rangle$.

The expected value of an observable Ω in a state $|\psi\rangle$

Since the $|e_i\rangle$'s are eigenvectors of Ω, applying Ω to each of them gives $\Omega|e_i\rangle = \lambda_i|e_i\rangle$. Therefore:

$$\begin{aligned}
\Omega|\psi\rangle &= \Omega(c_0|e_0\rangle + c_1|e_1\rangle + \ldots + c_{n-1}|e_{n-1}\rangle) && \text{by expanding } |\psi\rangle \text{ into its components} \\
&= \Omega c_0|e_0\rangle + \Omega c_1|e_1\rangle + \ldots + \Omega c_{n-1}|e_{n-1}\rangle && \text{since } \Omega \text{ is a linear operator} \\
&= c_0\Omega|e_0\rangle + c_1\Omega|e_1\rangle + \ldots + c_{n-1}\Omega|e_{n-1}\rangle && \text{since } \Omega \text{ is a linear operator} \\
&= c_0\lambda_0|e_0\rangle + c_1\lambda_1|e_1\rangle + \ldots + c_{n-1}\lambda_{n-1}|e_{n-1}\rangle && \text{since the } |e_i\rangle\text{'s are eigenvectors of } \Omega
\end{aligned}$$

Computing the inner product $\langle\psi|\,\Omega\,|\psi\rangle$ gives:

$$\begin{aligned}
\langle\psi|\,\Omega\,|\psi\rangle &= {c_0}^*c_0\lambda_0 + {c_1}^*c_1\lambda_1 + \ldots + {c_{n-1}}^*c_{n-1}\lambda_{n-1} && \text{since the amplitudes of } \langle\psi| \text{ are } {c_i}^* \\
&= |c_0|^2\lambda_0 + |c_1|^2\lambda_1 + \ldots + |c_{n-1}|^2\lambda_{n-1} && \text{since } {c_i}^*c_i = |c_i|^2
\end{aligned}$$

Since the $|c_i|^2$ values are each in the range 0–1, and together sum to 1, the above expression is simply the *weighted average* of the λ_i's, with each λ_i weighted by its coefficient $|c_i|^2$. The value $\langle\psi|\,\Omega\,|\psi\rangle$ thus represents the *expected value* or *mean value* of the λ_i's in the state $|\psi\rangle$, and is sometimes abbreviated as $\langle\Omega\rangle_\psi$. In other words, if we repeatedly observe Ω in the state $|\psi\rangle$, we will obtain a sequence of eigenvalues, the average value of which will approach $\langle\psi|\,\Omega\,|\psi\rangle$ more and more closely as the number of observations increases.

If $|\psi\rangle$ happens to be an eigenstate, say $|e_k\rangle$, then the amplitude of $|e_k\rangle$ in the expansion of $|\psi\rangle$ is 1, with all other amplitudes 0, which means that $\langle\psi|\,\Omega\,|\psi\rangle$ reduces to $|1|^2\lambda_k = \lambda_k$. In other words, the expected value of observing Ω in an eigenstate is exactly that eigenstate's associated eigenvalue.

In general, writing $\langle\phi|\,\Omega\,|\psi\rangle$ is equivalent to writing $\langle\phi\,|\,\Omega\psi\rangle$, with Ω applied to the right side of the inner product. If Ω is a hermitian operator, we can move it to the left side:

$$\begin{array}{rcll}
\langle\phi\,|\,\Omega\psi\rangle & = & \phi^\dagger(\Omega\psi) & \text{by rewriting } \langle A\,|\,B\rangle \text{ as } A^\dagger B \\
& = & (\phi^\dagger\Omega)\psi & \text{by associativity} \\
& = & (\phi^\dagger\Omega^\dagger)\psi & \text{since } \Omega \text{ is hermitian} \\
& = & (\Omega\phi)^\dagger\psi & \text{since } B^\dagger A^\dagger \text{ is equivalent to } (AB)^\dagger \\
& = & \langle\Omega\phi\,|\,\psi\rangle & \text{by rewriting } A^\dagger B \text{ as } \langle A\,|\,B\rangle
\end{array}$$

So applying a hermitian operator Ω to either side of an inner product $\langle\phi\,|\,\psi\rangle$ yields the same result:

$$\langle\phi\,|\,\Omega\psi\rangle = \langle\Omega\phi\,|\,\psi\rangle$$

In the special case when $\phi = \psi$:

$$\begin{array}{rcll}
\langle\psi\,|\,\Omega\psi\rangle & = & \langle\Omega\psi\,|\,\psi\rangle & \\
& = & \langle\psi\,|\,\Omega\psi\rangle^* & \quad\text{since } \langle A\,|\,B\rangle = \langle B\,|\,A\rangle^*
\end{array}$$

which implies that $\langle\psi\,|\,\Omega\psi\rangle$ must be real, because it is equal to its own conjugate. This is the same as $\langle\psi|\,\Omega\,|\psi\rangle$, the expected (or mean) value of observing Ω repeatedly in state $|\psi\rangle$. Thus the expected value of an observable is always a real number.

The effect of measurement

If a quantum system is in state $|\psi\rangle$ when we measure an observable Ω, the result obtained will always be one of the eigenvalues of Ω, and the state of the system will jump to one of the eigenstates. This "transition" is in general nondeterministic, and depends on the transition amplitudes for each eigenstate, which are just the coefficients c_i in the expansion of $|\psi\rangle$:

$$|\psi\rangle = c_0|e_0\rangle + c_1|e_1\rangle + \ldots + c_i|e_i\rangle + \ldots + c_{n-1}|e_{n-1}\rangle$$

Since $|\psi\rangle$ is normalized, the probability that the system will transition to eigenstate $|e_i\rangle$ is just the squared magnitude $|c_i|^2$ of the amplitude c_i. However, if $|\psi\rangle$ is instead written in the standard canonical basis as $[z_0, z_1, \ldots, z_{n-1}]^T$, we first need to change from the canonical basis to the eigenbasis of Ω by computing the amplitudes of the eigenbasis vectors using the inner product:

$c_0 = \langle e_0\,|\,\psi\rangle$
$c_1 = \langle e_1\,|\,\psi\rangle$
$\ldots$
$c_{n-1} = \langle e_{n-1}\,|\,\psi\rangle$

We can then rewrite the state vector in the new eigenbasis as $|\psi\rangle = c_0|e_0\rangle + c_1|e_1\rangle + \ldots + c_{n-1}|e_{n-1}\rangle$, and from there calculate the probability of observing eigenvalue λ_i as $|c_i|^2 = |\langle e_i\,|\,\psi\rangle|^2$.

Example: Spin state $|\textit{left}\rangle$

1. Express the state $|\textit{left}\rangle$ in terms of each of the three spin operator eigenbases.

   - For operator X, we compute the projection of $|\textit{left}\rangle$ onto the eigenvectors of X to find the amplitudes of $|\textit{right}\rangle$ and $|\textit{left}\rangle$:

     $$\langle \textit{right} \mid \textit{left}\rangle = \begin{bmatrix}\left(\frac{1}{\sqrt{2}}\right)^* & \left(\frac{1}{\sqrt{2}}\right)^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

     $$\langle \textit{left} \mid \textit{left}\rangle = \begin{bmatrix}\left(\frac{1}{\sqrt{2}}\right)^* & \left(\frac{-1}{\sqrt{2}}\right)^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

     So $|\textit{left}\rangle = 0\,|\textit{right}\rangle + 1\,|\textit{left}\rangle$

   - For operator Y, we compute the projection of $|\textit{left}\rangle$ onto the eigenvectors of Y to find the amplitudes of $|\textit{in}\rangle$ and $|\textit{out}\rangle$:

     $$\langle \textit{in} \mid \textit{left}\rangle = \begin{bmatrix}\left(\frac{1}{\sqrt{2}}\right)^* & \left(\frac{i}{\sqrt{2}}\right)^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2} + \frac{i}{2} = \frac{1+i}{2}$$

     $$\langle \textit{out} \mid \textit{left}\rangle = \begin{bmatrix}\left(\frac{1}{\sqrt{2}}\right)^* & \left(\frac{-i}{\sqrt{2}}\right)^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2} - \frac{i}{2} = \frac{1-i}{2}$$

     So $|\textit{left}\rangle = \frac{1+i}{2}\,|\textit{in}\rangle + \frac{1-i}{2}\,|\textit{out}\rangle$

   - For operator Z, we compute the projection of $|\textit{left}\rangle$ onto the eigenvectors of Z to find the amplitudes of $|\textit{up}\rangle$ and $|\textit{down}\rangle$:

     $$\langle \textit{up} \mid \textit{left}\rangle = \begin{bmatrix}1^* & 0^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{1}{\sqrt{2}} \qquad \langle \textit{down} \mid \textit{left}\rangle = \begin{bmatrix}0^* & 1^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \frac{-1}{\sqrt{2}}$$

     So $|\textit{left}\rangle = \frac{1}{\sqrt{2}}\,|\textit{up}\rangle - \frac{1}{\sqrt{2}}\,|\textit{down}\rangle$

2. What are the probabilities of obtaining $+1$ or -1 if we measure Z in state $|\textit{left}\rangle$?

   - The probability p_1 of $+1$ is the squared magnitude of the amplitude of $|\textit{up}\rangle$: $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$
   - The probability p_2 of -1 is the squared magnitude of the amplitude of $|\textit{down}\rangle$: $|\frac{-1}{\sqrt{2}}|^2 = \frac{1}{2}$

3. What is the *expected value* of measuring Z in state $|\textit{left}\rangle$?

   - Weighted average of eigenvalues: $p_1 \cdot \lambda_1 + p_2 \cdot \lambda_2 \;=\; \frac{1}{2}\cdot(+1) \;+\; \frac{1}{2}\cdot(-1) \;=\; 0$
   - $\langle \textit{left}|\, Z\, |\textit{left}\rangle = \begin{bmatrix}\left(\frac{1}{\sqrt{2}}\right)^* & \left(\frac{-1}{\sqrt{2}}\right)^*\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\left(\frac{1}{\sqrt{2}}\right)^* & \left(\frac{-1}{\sqrt{2}}\right)^*\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix} = \frac{1}{2} - \frac{1}{2} = 0$