Chapter 2: Review of Linear Algebra

Complex conjugate

Let $z = a + bi$. The complex conjugate of z is $a - bi$ and is written z^* or $\overline{z}$

- geometric interpretation of $-z$
- geometric interpretation of z^*
- the conjugate operation z^* *complements* (*negates*) the phase of z ("the other slice of the 2π")
- $zz^* = z^*z = a^2 + b^2 = |z|^2 = \rho^2$
- z^2 is in general complex, but $z^*z = |z|^2$ is always real (and non-negative)
- $r^* = r$ for all real numbers r
- $(x + y + z + \ldots)^* = (x^* + y^* + z^* + \ldots)$

Vectors and vector spaces

flatland

x-ray vision

hypercubes

what is a "vector space"?

- just a collection of elements, which we'll call "points"
- there is a special element called the "origin"
- a "vector" tells how to reach a particular element starting from the origin
- people often use "vector" and "point" interchangeably

examples of vector spaces

- real vectors in 1-D, 2-D, 3-D, and higher dimensions
- corners of a hypercube
- complex number plane (1-D complex vector space $\equiv$ 2-D real vector space)
- complex vectors in 2, 3, and higher dimensions
- qubit states (2-D complex vector space $\equiv$ 4-D real vector space)

visualizing operations on vectors:

- inverse of a vector (negation)
- multiplication by a scalar (positive or negative)
- vector addition
- these operations all behave as expected for commutativity, associativity, and distributivity: $cV = Vc$, $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$, $c(V_1 + V_2) = cV_1 + cV_2$, etc.

"every complex vector space is also a real vector space" (example 2.2.2, page 35)

An example of a real vector space is the real number line. This is a 1-D real vector space. If you take any vector from this space and multiply it by any real scalar, the resulting vector is still "trapped" in the same space. There is no way to jump out of this space by multiplying one of its elements by a real scalar. But if we were to multiply the real vector 3 by a complex scalar, say i, we would end up with the vector $3i$, which no longer lies "inside" the real number line: in fact, it is perpendicular to it. Multiplying by i jumps us out of the space. So although the real number line is a real vector space, it is not a complex vector space.

Now consider the complex plane. This is a 1-D complex vector space. If you take any vector from this space and multiply it by any complex scalar, the resulting complex vector will still lie in the plane. You can't jump out of the plane by multiplying by a complex scalar. So the complex plane is a complex vector space. But it is also a real vector space, because if you multiply any complex vector by a real scalar, you'll still be somewhere in the complex plane.

length ("norm") of a complex vector $\|A\| = \sqrt{|a_0|^2 + |a_1|^2 + |a_2|^2} = \sqrt{a_0{}^* a_0 + a_1{}^* a_1 + a_2{}^* a_2}$

example: norm of $\begin{bmatrix} 3+4i \\ 2-i \end{bmatrix} = \sqrt{|3+4i|^2 + |2-i|^2} = \sqrt{(3+4i)(3-4i) + (2-i)(2+i)} = \sqrt{30}$

norm $\|A\|$ vs. magnitude $|z|$

linear combinations of vectors

linear dependence and independence

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -5 \\ 11 \end{bmatrix}$ are not linearly independent, because $v_3 = 4v_1 + 3v_2$

basis vectors

orthogonal basis

orthonormal basis

standard/canonical basis example: $\begin{bmatrix} -3 \\ 5 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Bra and ket notation

"ket": $|V\rangle = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x, y, z]^T$ "bra": $\langle V| = V^\dagger = \begin{bmatrix} x^* & y^* & z^* \end{bmatrix}$

"bra-ket": $\langle V_1 \,|\, V_2 \rangle$

The inner product

Let $|A\rangle = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [a_0, a_1, a_2]^T$ and $|B\rangle = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = [b_0, b_1, b_2]^T$.

$$\langle A \,|\, B \rangle = A^\dagger B = \begin{bmatrix} {a_0}^* & {a_1}^* & {a_2}^* \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = {a_0}^* b_0 + {a_1}^* b_1 + {a_2}^* b_2$$

Conjugate * has no effect when vector elements are real

Important property: $\langle A \,|\, B \rangle = \langle B \,|\, A \rangle^*$

Alternative form of condition (iii) on page 54:

- To move a scalar from the outside to the inside of an inner product:
  $c \cdot \langle A \,|\, B \rangle = \langle \overline{c} \cdot A \,|\, B \rangle = \langle A \,|\, c \cdot B \rangle$
- To move a scalar from the inside to the outside of an inner product:
  $\langle c \cdot A \,|\, B \rangle = \overline{c} \cdot \langle A \,|\, B \rangle$
  $\langle A \,|\, c \cdot B \rangle = c \cdot \langle A \,|\, B \rangle$

We can compute the norm of A (its size/magnitude) using the inner product:

$\langle A \,|\, A \rangle = {a_0}^* a_0 + {a_1}^* a_1 + {a_2}^* a_2 = |a_0|^2 + |a_1|^2 + |a_2|^2 = \|A\|^2$

so $\|A\| = \sqrt{\langle A \,|\, A \rangle}$

Notation: $\|A\|$ is typically used for vectors, and $|z|$ for numbers, but they mean the same thing: the length/size/magnitude of a quantity with "direction".

Since $\langle A \,|\, B \rangle = \langle B \,|\, A \rangle^*$, we know that $\langle A \,|\, A \rangle = \langle A \,|\, A \rangle^*$, which means that $\langle A \,|\, A \rangle$ must be real.

$\langle A \,|\, B \rangle = 0$ means that A and B are orthogonal.

Basis vectors

If $\{B_1, B_2, \ldots, B_n\}$ is any set of n orthonormal basis vectors, then a vector V can be decomposed into its basis components by using the inner product to project V onto each of the basis vectors:

$$V = \langle B_1 \,|\, V \rangle\, B_1 + \langle B_2 \,|\, V \rangle\, B_2 + \ldots + \langle B_n \,|\, V \rangle\, B_n$$

That is, the inner products give the coefficients or "amplitudes" of the basis vectors.

Example: consider two different orthonormal basis sets A and B:

$$|A_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |A_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|B_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } |B_2\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and a vector $V = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

The description of V in the "language" of basis set A is expressed as a linear combination of the building blocks $|A_1\rangle$ and $|A_2\rangle$. To calculate the coefficient or "amplitude" of each building block, we use the inner product:

$$\text{amplitude of } |A_1\rangle = \langle A_1 \,|\, V \rangle = \begin{bmatrix} 1^* & 0^* \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = 1 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{\sqrt{3}}{2} = \tfrac{1}{2}$$

$$\text{amplitude of } |A_2\rangle = \langle A_2 \,|\, V \rangle = \begin{bmatrix} 0^* & 1^* \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = 0 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{2}$$

so $V = \frac{1}{2}\,|A_1\rangle + \frac{\sqrt{3}}{2}\,|A_2\rangle$

$= 0.5\,|A_1\rangle + 0.866\,|A_2\rangle$

To express V in the language of basis set B, we calculate B's amplitudes in the same way:

$$\text{amplitude of } |B_1\rangle = \langle B_1 \,|\, V \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}^* & \frac{1}{\sqrt{2}}^* \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \tfrac{1}{\sqrt{2}} \cdot \tfrac{1}{2} + \tfrac{1}{\sqrt{2}} \cdot \tfrac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}} = 0.966$$

$$\text{amplitude of } |B_2\rangle = \langle B_2 \,|\, V \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}^* & -\frac{1}{\sqrt{2}}^* \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \tfrac{1}{\sqrt{2}} \cdot \tfrac{1}{2} - \tfrac{1}{\sqrt{2}} \cdot \tfrac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2\sqrt{2}} = -0.259$$

so $V = 0.966\,|B_1\rangle - 0.259\,|B_2\rangle$

Example: RGB colors

Imagine a 3-D real vector space with elements $[r, g, b]^T$, where $0 \leq r \leq 1$, $0 \leq g \leq 1$, and $0 \leq b \leq 1$.

$\mathit{red} = [1, 0, 0]^T$
$\mathit{green} = [0, 1, 0]^T$
$\mathit{blue} = [0, 0, 1]^T$
$\mathit{magenta} = [1, 0, 1]^T$
$\mathit{yellow} = [1, 1, 0]^T$
$\mathit{white} = [1, 1, 1]^T$
$\mathit{black} = [0, 0, 0]^T$
$\mathit{gray} = [0.5, 0.5, 0.5]^T$

Basis vectors:

$|R\rangle = \mathit{red} = [1, 0, 0]^T$
$|G\rangle = \mathit{green} = [0, 1, 0]^T$
$|B\rangle = \mathit{blue} = [0, 0, 1]^T$

Arbitrary color vector: $V = r|R\rangle + g|G\rangle + b|B\rangle$

Intensity is the vector length $\|V\|$, but suppose we don't care about intensity, only hue.

RGB "spheramid" restricts color intensity to $\|V\| = 1$, so that $|r|^2 + |g|^2 + |b|^2 = 1$

So no black or white allowed! Only red, green, blue, and linear combinations thereof. We can still make magenta, yellow, orange, etc. (and even gray), but they just won't be as intense as they were in the RGB cube.

$\mathit{red} = [1, 0, 0]^T$
$\mathit{green} = [0, 1, 0]^T$
$\mathit{blue} = [0, 0, 1]^T$
$\mathit{magenta} = [\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}]^T$
$\mathit{yellow} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0]^T$
$\mathit{cyan} = [0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^T$
$\mathit{gray} = [\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]^T$

$\mathit{turquoise} = [0.18, 0.72, 0.67]^T = 0.18\,|R\rangle + 0.72\,|G\rangle + 0.67\,|B\rangle$

Combining two vector spaces with the Cartesian product $\times$

- Elements of vector space A specify *my* shirt color. A is an "RGB spheramid" space ($\frac{1}{8}$ of the surface of a 3-D sphere), with basis vectors $|R_1\rangle$, $|G_1\rangle$, and $|B_1\rangle$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \text{amount of } |R_1\rangle \\ \text{amount of } |G_1\rangle \\ \text{amount of } |B_1\rangle \end{bmatrix} \qquad \text{Example: } \begin{bmatrix} 0.18 \\ 0.72 \\ 0.67 \end{bmatrix} = 0.18\,|R_1\rangle + 0.72\,|G_1\rangle + 0.67\,|B_1\rangle$$

- Elements of vector space B specify *your* shirt color. B is a separate "RGB spheramid" space, with basis vectors $|R_2\rangle$, $|G_2\rangle$, and $|B_2\rangle$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \text{amount of } |R_2\rangle \\ \text{amount of } |G_2\rangle \\ \text{amount of } |B_2\rangle \end{bmatrix} \qquad \text{Example: } \begin{bmatrix} 0.50 \\ 0.84 \\ 0.22 \end{bmatrix} = 0.50\,|R_2\rangle + 0.84\,|G_2\rangle + 0.22\,|B_2\rangle$$

The *Cartesian product* or *cross product* $A \times B$ (read "A cross B") represents the combination of my shirt color and your shirt color together. It is a 6-D real vector space. In general, if A is an m-dimensional space and B is an n-dimensional space, the combined vector space $A \times B$ is an $(m+n)$-dimensional space.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \times \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \text{amount of } |R_1\rangle \\ \text{amount of } |G_1\rangle \\ \text{amount of } |B_1\rangle \\ \text{amount of } |R_2\rangle \\ \text{amount of } |G_2\rangle \\ \text{amount of } |B_2\rangle \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.72 \\ 0.67 \\ 0.50 \\ 0.84 \\ 0.22 \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.72 \\ 0.67 \end{bmatrix} \times \begin{bmatrix} 0.50 \\ 0.84 \\ 0.22 \end{bmatrix}$$

This vector, like all vectors from $A \times B$, can be trivially decomposed into the Cartesian product of a vector from A and a vector from B. Notice that the length of the vector is *not* equal to 1.

Both wearing red $= [1, 0, 0, 1, 0, 0]^T$

Both wearing green $= [0, 1, 0, 0, 1, 0]^T$

Both wearing blue $= [0, 0, 1, 0, 0, 1]^T$

Me wearing red and you wearing green $= [1, 0, 0, 0, 1, 0]^T$

Both wearing magenta $= [0.71, 0, 0.71, 0.71, 0, 0.71]^T$

Both wearing gray $= [0.58, 0.58, 0.58, 0.58, 0.58, 0.58]^T$

Me wearing yellow and you wearing red $= [0.71, 0.71, 0, 1, 0, 0]^T$

Me wearing yellow and you wearing gray $= [0.71, 0.71, 0, 0.58, 0.58, 0.58]^T$
…etc.

Combining two vector spaces with the tensor product $\otimes$

The *tensor product* $A \otimes B$ is a 9-dimensional real vector space. The tips of the 9 basis vectors all touch the curved 8-D "surface" of a 9-D hypersphere. This surface is $\frac{1}{512}$ the total surface of the hypersphere.

In general, if A is an m-dimensional space and B is an n-dimensional space, the combined vector space $A \otimes B$ is $(m \cdot n)$-dimensional.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0b_0 \\ a_0b_1 \\ a_0b_2 \\ a_1b_0 \\ a_1b_1 \\ a_1b_2 \\ a_2b_0 \\ a_2b_1 \\ a_2b_2 \end{bmatrix} = \begin{bmatrix} \text{joint amount of } |R_1\rangle \text{ and } |R_2\rangle \\ \text{joint amount of } |R_1\rangle \text{ and } |G_2\rangle \\ \text{joint amount of } |R_1\rangle \text{ and } |B_2\rangle \\ \text{joint amount of } |G_1\rangle \text{ and } |R_2\rangle \\ \text{joint amount of } |G_1\rangle \text{ and } |G_2\rangle \\ \text{joint amount of } |G_1\rangle \text{ and } |B_2\rangle \\ \text{joint amount of } |B_1\rangle \text{ and } |R_2\rangle \\ \text{joint amount of } |B_1\rangle \text{ and } |G_2\rangle \\ \text{joint amount of } |B_1\rangle \text{ and } |B_2\rangle \end{bmatrix} = \begin{bmatrix} 0.09 \\ 0.15 \\ 0.04 \\ 0.36 \\ 0.60 \\ 0.16 \\ 0.34 \\ 0.56 \\ 0.15 \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.72 \\ 0.67 \end{bmatrix} \otimes \begin{bmatrix} 0.50 \\ 0.84 \\ 0.22 \end{bmatrix}$$

This particular vector is *separable*, because it can be decomposed into the tensor product of a vector from A and a vector from B. However, most vectors from $A \otimes B$ *cannot* be decomposed in this way. Such vectors are *entangled*. The 9-dimensional space $A \otimes B$ is vastly larger than the 6-dimensional space $A \times B$. This is where the quantum phenomenon of *entanglement* comes from.

Both wearing red $= [1, 0, 0, 0, 0, 0, 0, 0, 0]^T$

Both wearing green $= [0, 0, 0, 0, 1, 0, 0, 0, 0]^T$

Both wearing blue $= [0, 0, 0, 0, 0, 0, 0, 0, 1]^T$

Me wearing red and you wearing green $= [0, 1, 0, 0, 0, 0, 0, 0, 0]^T$

Both wearing magenta $= [0.5, 0, 0.5, 0, 0, 0, 0.5, 0, 0.5]^T$

Both wearing gray $= [0.33, 0.33, 0.33, 0.33, 0.33, 0.33, 0.33, 0.33, 0.33]^T$

Me wearing yellow and you wearing red $= [0.71, 0, 0, 0.71, 0, 0, 0, 0, 0]^T$

Me wearing yellow and you wearing gray $= [0.41, 0.41, 0.41, 0.41, 0.41, 0.41, 0, 0, 0]^T$
. . .etc.

Notice that the length of a vector from $A \otimes B$ is always equal to 1.

Rules for tensor products

- $|A\rangle \otimes |B\rangle$ can be abbreviated as $|AB\rangle$
- $\otimes$ does not commute: $|A\rangle \otimes |B\rangle \neq |B\rangle \otimes |A\rangle$
- $\otimes$ distributes over $+$:

$$|A\rangle \otimes \Big(|B\rangle + |C\rangle\Big) = |A\rangle \otimes |B\rangle + |A\rangle \otimes |C\rangle$$
$$\Big(|A\rangle + |B\rangle\Big) \otimes |C\rangle = |A\rangle \otimes |C\rangle + |B\rangle \otimes |C\rangle$$

- scalar multiplication "semi-distributes" over $\otimes$:

$$\alpha\Big(|A\rangle \otimes |B\rangle\Big) = \alpha|A\rangle \otimes |B\rangle = |A\rangle \otimes \alpha|B\rangle$$

Example: a separable state

"me wearing yellow" $\otimes$ "you wearing red"

$$= \begin{bmatrix}0.71\\0.71\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \left(0.71\begin{bmatrix}1\\0\\0\end{bmatrix} + 0.71\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \otimes \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \Big(\; 0.71 \text{ “me wearing red” } + 0.71 \text{ “me wearing green” } \;\Big) \otimes \text{“you wearing red”}$$

$$= \Big(0.71\,|R_1\rangle + 0.71\,|G_1\rangle\Big) \otimes |R_2\rangle$$

$$= 0.71\,|R_1\rangle \otimes |R_2\rangle + 0.71\,|G_1\rangle \otimes |R_2\rangle$$

$$= 0.71\Big(|R_1\rangle \otimes |R_2\rangle\Big) + 0.71\Big(|G_1\rangle \otimes |R_2\rangle\Big)$$

$$= 0.71\,|R_1R_2\rangle + 0.71\,|G_1R_2\rangle$$

$$= 0.71 \text{ “me wearing red and you wearing red” } + 0.71 \text{ “me wearing green and you wearing red”}$$

$$= 0.71\left(\begin{bmatrix}1\\0\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\\0\end{bmatrix}\right) + 0.71\left(\begin{bmatrix}0\\1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

$$= [0.71, 0, 0, 0.71, 0, 0, 0, 0, 0]^T$$

Example: an entangled state

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_0 b_2 \\ a_1 b_0 \\ a_1 b_1 \\ a_1 b_2 \\ a_2 b_0 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0.71 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.71 \end{bmatrix}$$

$a_0 b_0 = 0.71$, so a_0 cannot be 0, and $a_2 b_2 = 0.71$, so b_2 cannot be 0.

But $a_0 b_2 = 0$, so at least one of a_0 and b_2 must be 0. This is a contradiction, so there is no way to choose values to make the above relationship true, and thus no way to write this 9-D state as the tensor product of two 3-D states.

Thus, the state $[0.71, 0, 0, 0, 0, 0, 0, 0, 0.71]^T$ is entangled.

It can, however, be written as a *linear combination* of tensor products of 3-D states, since in general any 9-D state can be expressed as a combination of basis states:

$$\alpha \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \beta \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = 0.71 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0.71 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0.71 \, |R_1 R_2\rangle + 0.71 \, |B_1 B_2\rangle = \begin{bmatrix} 0.71 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.71 \end{bmatrix}$$

This entangled state is a linear combination of the 9-D basis states "both of us wearing red" and "both of us wearing blue", with amplitudes 0.71 and 0.71. Likewise, the separable state $[0.71, 0, 0, 0.71, 0, 0, 0, 0, 0]^T$ — "me wearing yellow and you wearing red" — is a linear combination of the 9-D basis states "both of us wearing red" and "me wearing green and you wearing red":

$$\alpha \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \beta \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = 0.71 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0.71 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0.71 \, |R_1 R_2\rangle + 0.71 \, |G_1 R_2\rangle = \begin{bmatrix} 0.71 \\ 0 \\ 0 \\ 0.71 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrices

formal structure

- dimensions of an $M \times N$ matrix
- 0-based indexing of elements by row and column number

matrices as operations on vectors

- we will almost always use *square* matrices
- square matrices of complex numbers represent (1) *operations* on quantum systems, and (2) *observable properties* of quantum systems
- how to multiply an $M \times N$ matrix and an N-vector
- geometric interpretation
- examples

potential point of confusion: we might think of a length-N vector as a “1-dimensional array” in a programming language, but it represents an *N-dimensional* quantum system. Likewise, we might think of an $N \times N$ matrix as a “2-dimensional array”, but it represents an operation on an *N-dimensional* quantum system.

identity matrix

transforming and combining matrices

- addition
- scalar multiplication
- matrix multiplication
- matrix multiplication *does not commute* (source of Heisenberg’s uncertainty principle)
- transpose M^T
- conjugate $M^* = \overline{M}$
- adjoint $M^\dagger = (M^T)^* = (M^*)^T$

important properties:

- $(AB)^* = A^* B^*$
- $(AB)^T = B^T A^T$
- $(AB)^\dagger = B^\dagger A^\dagger$

Matrix multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \cdot 10 + 2 \cdot 11 + 3 \cdot 12 \\ 4 \cdot 10 + 5 \cdot 11 + 6 \cdot 12 \\ 7 \cdot 10 + 8 \cdot 11 + 9 \cdot 12 \end{bmatrix} = \begin{bmatrix} 10 + 22 + 36 \\ 40 + 55 + 72 \\ 70 + 88 + 108 \end{bmatrix} = \begin{bmatrix} 68 \\ 167 \\ 266 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -i \\ 3i & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \begin{bmatrix} 2(1+i) + (-i)(1-i) \\ 3i(1+i) + 1(1-i) \end{bmatrix} = \begin{bmatrix} 2 + 2i - i - 1 \\ 3i - 3 + 1 - i \end{bmatrix} = \begin{bmatrix} 1+i \\ -2+2i \end{bmatrix}$$

Linearity

if you have two vectors V_1 and V_2, and an operation M:

you can *combine the vectors first* and then apply the operation to the resulting vector, or you can *apply the operation first* to each vector individually, and then combine the resulting vectors:

$M(V_1 + V_2) = MV_1 + MV_2$

you can *scale the vector first* and then apply the operation to the resulting vector, or you can *apply the operation first* and then scale the resulting vector:

$M(cV) = c(MV)$

Eigenvectors and eigenvalues

Example of a matrix $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ operating on various vectors:

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 3

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 2

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$\begin{bmatrix} 5 \\ 10 \end{bmatrix}$ is an eigenvector of A with eigenvalue 2