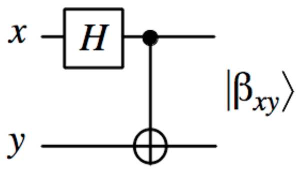
Bell states



$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \qquad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$H \otimes I = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$BELL = CNOT \star (H \otimes I) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

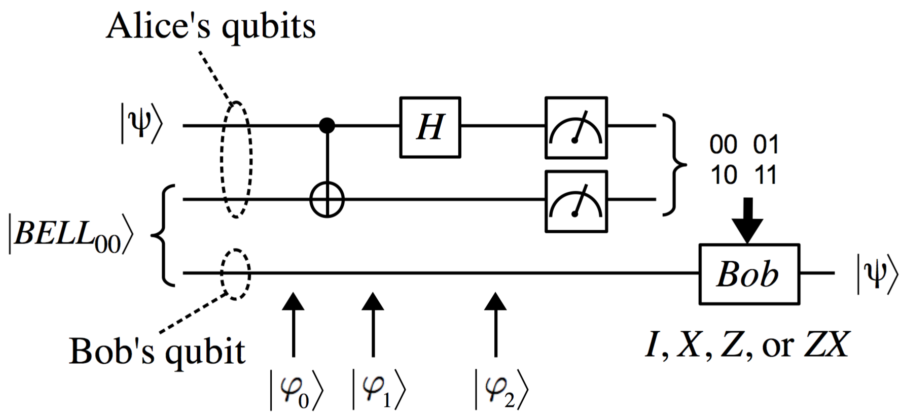
$$BELL|00\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{(column 0)}$$

$$BELL|00\rangle = |\beta_{00}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \qquad BELL|01\rangle = |\beta_{01}\rangle = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

$$BELL|10\rangle = |\beta_{10}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \qquad BELL|11\rangle = |\beta_{11}\rangle = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

These entangled states are also known as *EPR states* or *EPR pairs*.

Quantum teleportation



Qubit state to be teleported to Bob: $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$

Alice and Bob's EPR pair: $|BELL_{00}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$$|\varphi_0\rangle = |\psi\rangle \otimes |BELL_{00}\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\beta}{\sqrt{2}} \end{bmatrix} \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix} \quad = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

$$|\varphi_1\rangle = (CNOT \otimes I)|\varphi_0\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\beta}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ 0 \\ \frac{\beta}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix}$$

$$|\varphi_2\rangle = (H \otimes I \otimes I)|\varphi_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ 0 \\ \frac{\beta}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{2} \\ \frac{\beta}{2} \\ \frac{\beta}{2} \\ \frac{\alpha}{2} \\ \frac{\alpha}{2} \\ \frac{-\beta}{2} \\ \frac{-\beta}{2} \\ \frac{\alpha}{2} \end{bmatrix} \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix}$$

$$= \tfrac{1}{2}\Big[\alpha|000\rangle + \beta|001\rangle + \beta|010\rangle + \alpha|011\rangle + \alpha|100\rangle - \beta|101\rangle - \beta|110\rangle + \alpha|111\rangle\Big]$$

In general, $c|xyz\rangle$ can be rewritten as $c\Big(|x\rangle \otimes |y\rangle \otimes |z\rangle\Big) = |x\rangle \otimes |y\rangle \otimes c|z\rangle = |xy\rangle \otimes c|z\rangle$

So we can rewrite $|\varphi_2\rangle$ as:

$$\tfrac{1}{2}\Big[|00\rangle \otimes \alpha|0\rangle + |00\rangle \otimes \beta|1\rangle + |01\rangle \otimes \beta|0\rangle + |01\rangle \otimes \alpha|1\rangle + |10\rangle \otimes \alpha|0\rangle + |10\rangle \otimes -\beta|1\rangle + |11\rangle \otimes -\beta|0\rangle + |11\rangle \otimes \alpha|1\rangle\Big]$$

Since $A \otimes B + A \otimes C = A \otimes (B + C)$, we can rewrite the above subterms as follows:

$$|00\rangle \otimes \alpha|0\rangle + |00\rangle \otimes \beta|1\rangle \quad \rightarrow \quad |00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$$

$$|01\rangle \otimes \beta|0\rangle + |01\rangle \otimes \alpha|1\rangle \quad \rightarrow \quad |01\rangle \otimes (\beta|0\rangle + \alpha|1\rangle)$$

$$|10\rangle \otimes \alpha|0\rangle + |10\rangle \otimes -\beta|1\rangle \quad \rightarrow \quad |10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)$$

$$|11\rangle \otimes -\beta|0\rangle + |11\rangle \otimes \alpha|1\rangle \quad \rightarrow \quad |11\rangle \otimes (-\beta|0\rangle + \alpha|1\rangle)$$

So we can rewrite $|\varphi_2\rangle$ as:

$$\tfrac{1}{2}|00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + \tfrac{1}{2}|01\rangle \otimes (\beta|0\rangle + \alpha|1\rangle) + \tfrac{1}{2}|10\rangle \otimes (\alpha|0\rangle - \beta|1\rangle) + \tfrac{1}{2}|11\rangle \otimes (-\beta|0\rangle + \alpha|1\rangle)$$

Now Alice measures her two qubits. The probability of obtaining 00 is $\left|\frac{1}{2}\right|^2 = \frac{1}{4}$, which is also the probability of obtaining 01, 10, or 11, respectively. After the measurement, the state of the third qubit (Bob's) must be:

Alice's measurement	State of Bob's qubit
00	$\alpha\|0\rangle + \beta\|1\rangle$
01	$\beta\|0\rangle + \alpha\|1\rangle$
10	$\alpha\|0\rangle - \beta\|1\rangle$
11	$-\beta\|0\rangle + \alpha\|1\rangle$

Alice now communicates the result of her measurement to Bob, which requires her to send Bob just two classical bits of information: 00, 01, 10, or 11. This communication is limited by the speed of light, so if Bob is very far away, he may have to wait a long time to receive Alice's bits.

Once Bob receives this information, he performs the following action on his qubit:

Bits received	Bob's action	Resulting qubit state
00	Do nothing	$\alpha\|0\rangle + \beta\|1\rangle = \|\psi\rangle$
01	Apply X to his qubit	$X\Big(\beta\|0\rangle + \alpha\|1\rangle\Big) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \alpha\|0\rangle + \beta\|1\rangle = \|\psi\rangle$
10	Apply Z to his qubit	$Z\Big(\alpha\|0\rangle - \beta\|1\rangle\Big) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} = \alpha\|0\rangle + \beta\|1\rangle = \|\psi\rangle$
11	Apply ZX to his qubit	$ZX\Big(-\beta\|0\rangle + \alpha\|1\rangle\Big) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\beta \\ \alpha \end{bmatrix} = \alpha\|0\rangle + \beta\|1\rangle = \|\psi\rangle$

So Bob is able to recover the original qubit state $|\psi\rangle$ in all cases.

This qubit state, potentially encoding an infinite amount of quantum information in its two complex amplitudes, has been "teleported" from Alice to Bob using only a single EPR pair and a classical communication channel of very low bandwidth (two bits). However, the qubit state has not been copied, because Alice no longer possesses her qubit $|\psi\rangle$, which became a classical bit after she measured it. So there was no violation of the No-Cloning Theorem. Furthermore, although Bob's new qubit state has exactly the same complex amplitudes as Alice's original qubit, measuring the qubit can only provide Bob with a single bit of information.