Formalized Number Theory

- Statements of formalized number theory refer to **numbers**

"6 is even" $\exists a:(SS0\cdot a)=SSSSSS0$

"6 is odd" $\sim\exists a:(SS0\cdot a)=SSSSSS0$

"$b \geq 3$" $\exists a:SSSa=b$

"b is prime" $<< \sim b=0 \wedge \sim b=S0 > \wedge \sim\exists c:\exists d:(SSc\cdot SSd)=b >$

"b is a power of 2" $< b=S0 \vee < \exists a:(SS0\cdot Sa)=b \wedge \sim\exists c:\exists d:(SSSc\cdot Sd)=b >$

"there are infinitely many prime numbers" $\forall d:\exists e:\sim\exists b:\exists c:(d+Se)=(SSb\cdot SSc)$

"1729 is expressible as the sum of two cubes"

etc...

Formalized Number Theory

- Statements of formalized number theory refer to **numbers**
- How could a statement possibly refer to another **statement**?
- Answer: by assigning a **code number** to every statement!
- This number is called the statement's **Gödel number**
- Via Gödel numbering, a mathematical statement can refer to another mathematical statement “in code” – or even **to itself!**

Formalized Number Theory

- This was Gödel's stroke of genius
- “Axioms” and “theorems” are really just **numbers** in disguise
- Deriving new theorems from old theorems by applying the formal system's “rules” is really just computing **new numbers** from **old numbers** using complex arithmetical operations
- These **operations on numbers** can themselves be described in the formalized language of number theory

Gödel's Method

- Epimenides paradox:
  - *"I am lying"*
  - *"This statement is false"*
- Gödel's construction:
  - *"This statement is unprovable"*
  - *"The number N cannot be derived using the system's rules"* where N is the Gödel number of that very statement
- What if we could **derive it** using the system's rules? It would be **false**, so the system would be **inconsistent**
- What if we could **not derive it** using the system's rules? It would be **true**, so the system would be **incomplete**

Gödel's Incompleteness Theorem

All **consistent** axiomatic formalizations of mathematics are **incomplete**

Kurt Gödel (1906-78)

- For any formal mathematical system capable of representing the natural numbers, there exist **true statements** that can **never be proved** by the system

- Provability is a **weaker notion** than truth

Gödel-Numbering the MIU-System

<u>Rules</u>:

1. From x**I** make x**IU**
2. From **M**x make **M**xx
3. Replace **III** by **U**
4. Drop **UU**

<u>Axiom</u>:

MI

<u>Encoding Scheme</u>:

M = 3
I = 1
U = 0

<u>Derivation</u>:

MI	axiom	31
MII	rule 2	311
MIIII	rule 2	31111
MIIIIU	rule 1	311110
MIUU	rule 3	3100

Gödel-Numbering the MIU-System

We can now rephrase statements about the MIU-system as statements about numbers!

- "**MIUU** is a theorem of the MIU-system"

  = "3100 is a MIU-number"

- "**MU** is not a theorem of the MIU-system"

  = "30 is not a MIU-number"

TNT String	Interpretation
<< ~b=0 ^ ~b=S0 > ^ ~∃c:∃d:(SSc·SSd)=b >	"*b* is a prime number"
a *much* more complicated string of TNT	"*b* is a MIU-number" "*b* is a theorem of the MIU-system"
above string with **b** replaced by $\underbrace{\mathbf{SSSSSSS...S0}}_{30\ \mathbf{S}\text{'s}}$	"30 is a MIU-number" (MUMON) "**MU** is a theorem of the MIU-system"
an *insanely* complicated string of TNT	"*b* is a TNT-number" "*b* is a theorem of the TNT-system" "*b* is a true statement about numbers"
(S 0 + S 0) = S S 0 362,123,666,112,123,666,323,111,123,123,666	a "true" number
S 0 = 0 123,666,111,666	a "false" number

TNT String	Interpretation
(S 0 + S 0) = S S 0 362,123,666,112,123,666,323,111,123,123,666	a “true” number
S 0 = 0 123,666,111,666	a “false” number
~ statement about the number 123,666,111,666	“123,666,111,666 is not a TNT-number” “**S0=0** is not derivable in TNT” “**S0=0** is not a theorem of TNT”
~ statement about the number *G* (where *G* = the Gödel number of the above string)	“*G* is not a TNT-number” “this string is not derivable in TNT” “this string is not a theorem of TNT”

<u>TNT String</u>	<u>Interpretation</u>
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~ statement about the number *G* (where *G* = the Gödel number of the above string)	“*G* is not a TNT-number” “this string is not derivable in TNT” “this string is not a theorem of TNT”
What if it **could** be derived?	Then it would be **false**!
What if it **could not** be derived?	Then it would be **true**!
What if its **negation** could be derived?	Then its negation would be **false**!

It is **undecidable** within TNT